Noncommutative flow equivalence

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Dynamical systems

A dynamical system on a space X is (X, G, φ) consists of a topological group G together with an action $\varphi\colon G \times X \to X$, that is, a continuous map such that φ_0 is the identity and $\varphi_s \circ \varphi_t = \varphi_{s+t}$.

C*-dynamical systems

We say that (A, G, α) is a C*-dynamical system on a C* algebra A if G is a locally compact group and $\alpha \colon G \to \mathrm{Aut}(A)$ is a continuous homomorphism.

Theorem:

If $A = C(X)$ is a commutative C*-algebra, there is a correspondence between the dynamical systems on X and those on A.

Definition

Let (A, G, ϕ) be a dynamical system, that is, A is a C* algebra, G is a locally compact group and ϕ is an action of G on A.

If $T: X \to X$ is a homeomorphism, then $\alpha: C(X) \to C(X)$ defined by $\alpha(f) = f \circ T^{-1}$ is an automorphism of $C(X)$.

Hence, given a homeomorphism $T: X \to X$ one obtains a dynamical system $(C(X), \mathbb{Z}, \alpha)$.

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C* crossed products

Let (X, G, α) be a dynamical system. If $f\colon G \to A$ is continuous and has compact support, define

$$\|f\|_1 = \int_G \|f(s)\| \mathrm{d}\mu(s).$$

Call $L^1(G, A)$ the completion of such functions with this norm.

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C* crossed products

Addition in $L^1(G, A)$ is pointwise. Consider the product

$$(f * h)(s) = \int_G f(t)\alpha_t\left(h(t^{-1}s)\right)\, d\mu(t).$$

And a convolution

$$f^*(s) = \alpha_s(f(s^{-1})^*).$$

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C* crossed product

We call the crossed product of the dynamical system (A, G, α), denoted $C^*(A, G, \alpha)$, the completion of the algebra $L^1(G, A)$ with respect a suitable norm.

A theorem of Giordano, Putnam and Skau

In 1995, Giordano, Putnam and Skau tried to prove analogous results to those of Dye for (topological) dynamical system. They only succeeded for Cantor minimal systems.

Relations on dynamical systems ($G = \mathbb{Z}$)

We say that (X_1, ϕ_1) and (X_2, ϕ_2) are conjugate if there is a homeomorphism $F\colon X_1 \to X_2$ such that $F \circ \phi_1 = \phi_2 \circ F$.

We say that (X_1, ϕ_1) and (X_2, ϕ_2) are flip conjugate if (X_1, ϕ_1) is conjugate to either (X_2, ϕ_2) or to (X_2, ϕ_2^{-1}).

Relations on dynamical systems ($G = \mathbb{Z}$)

We say that (X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent if there is a homeomorphism (orbit map) $F\colon X_1 \to X_2$ such that $F(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(F(x))$ for all $x \in X_1$. Call orbit cocycles $n(x)$ and $m(x)$ the functions such that $F(\phi_1(x)) = \phi_2^{n(x)}(F(x))$ and $\phi_2(F(x)) = F(\phi_1^{m(x)}(x))$.

We say that (X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent if the is an orbit map F with cocicles admiting at most one point of discontinuity.

Theorem 1.

(X_1, ϕ_1) and (X_2, ϕ_2) are flip equivalent if and only if $C^*(X_1, \phi_1) \cong C^*(X_2, \phi_2)$ via an isomorphisms mapping $C(X_1)$ onto $C(X_2)$.

Theorem 2.

(X_1, ϕ_1) and (X_2, ϕ_2) are strong orbit equivalent if and only if $C^*(X_1, \phi_1) \cong C^*(X_2, \phi_2)$ are isomorphic.

Let (X, ϕ) be a dynamical system. The suspension of (X, ϕ) is a continuous flow (Y, T) where

$$Y = X \times [0,1]/(x,1) \sim (\phi(x), 0).$$

and

$$T^t([x,s]) = [x, s+t].$$

Flow equivalence

(X_1, ϕ_1) and (X_2, ϕ_2) are flow equivalent if their suspension flows (Y_1, T_1) and (Y_2, T_2) are topologically equivalent, that is, there is a homeomorphism $Y_1 \to Y_2$ which maps each or orbit of T_1 to an orbit of T_2, preserving orientation.

Banach Algebras

A Banach algebra A is a complete normed algebra.

We assume all our algebras are unital.

Examples:

- If X is compact Hausdorff then $C(X) = \{f\colon X \to \mathbb{C}\}$ with norm $\|\|_\infty$.
- More general, if A is a Banach algebra then $C(X, A) = \{f\colon X \to A\}$.
- Any C*- algebra

Mapping torus

Let $\alpha \in \mathrm{Aut}(A)$. We define

$$T_\alpha(A) = \Big\{ f\colon [0,1] \to A \mid f(1) = \alpha\,(f(0)) \Big\} \subset C([0,1], A).$$

It naturally induces a dynamical system $(T_\alpha(A), \mathbb{R}, \phi)$ given by

$$\phi_t(f)(s) = f(s+t) = \alpha^{[s+t]}\Big(f\,(\{s+t\})\Big)$$

Mapping torus

If (X, ϕ) is a dynamical system and (Y, T) its suspension flow, then $C(Y)$ is the mapping torus of $(C(X), \alpha)$, where $\alpha(f) = f \circ \phi^{-1}$.

Mapping torus

We say that $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ are conjugate if there is an isomorphism $\gamma\colon A \to B$ such that $\beta \circ \gamma = \gamma \circ \alpha$.
We say α and β are flip conjugate if α is conjugate to either β or β^{-1}.

Mapping torus

If α and β are flip conjugate then $T_\alpha(A)$ is isomorphic to $T_\beta(B)$

Proof: If γ is the conjugacy then $h\colon T_\alpha(A) \to T_{\beta^{\pm 1}}(B)$ given $h(f)(t) = \gamma(f(t))$ does the job. In the case of conjugacy with β^{-1}, verify that $r\colon T_\beta(B) \to T_{\beta^{-1}}(B)$ given by $r(f)(t) = f(1-t)$ is an isomorphism.

Mapping torus

¿What if $T_\alpha(A)$ is isomorphic to $T_\beta(B)$?

α and β are not necessary flip conjugate.

Example: Homeomorphisms of X which are flow equivalent but not flip conjugate.

Note: Isomorphism of mapping tori must mean some kind of noncommutative flow equivalence of (A, α) and (B, β).

Result

But true if A is simple (and unital).

Theorem 5.1 (IO, Perez-Ramirez).

Let A and B be a unital simple Banach algebras and let α and β be automorphisms of A and B, respectively. The mapping torus $T_{\alpha}(A)$ and $T_{\beta}(B)$ are isomorphic if and only if α and β are flip conjugate.

Proof.

Actually, α and β are flip conjugate if the isomorphism ϕ between their mapping torus maps maximal ideals $\ker \mathrm{ev}_t$ into maximal ideals of the same form, where $\mathrm{ev}_t \colon T_\alpha(A) \to A$ denotes evaluation at t, so that

$$\ker \mathrm{ev}_t = \left\{ f \in T_\alpha(A) \colon f(t) = 0 \right\}$$

And use same notation for evaluation in $T_\beta(B)$.

Maximal ideals

Work of de B. Yood (1951) and subsequent works (for instance W. J. Hery (1976), M. Abel and M. Abtahi (2013)) establish conditions for a bijection ($\mathcal{M}$ denote the set of maximal ideals)

$$h\colon X \times \mathcal{M}(A) \longrightarrow \mathcal{M}\left(C(X, A)\right)$$

given by

$$h(x, M) = \{f \in C(X, A)\colon f(x) \in M\}$$

Maximal ideals

Note that $T_{\mathrm{Id}}(A) = C\big(S^1, A\big)$ and in general $T_\alpha(A)$ is a subalgebra de $C\big([0,1], A\big)$.

Also, if $a \in A$ then the function $f(t) = (1-t)a + t\,\alpha(a)$ belongs to $T_\alpha(A)$ and $f(0) = a$.

An ideal I of $T_\alpha(A)$ is called fixed if

$$\bigcap_{f \in I} \{t \in [0,1] \colon f(t) \text{ not invertible}\} \neq \emptyset$$

Maximal ideals

Lemma 3.

Let $\mathcal{M}$ be any ideal of A, $t \in [0,1]$ and $a \in A$.

1. *There is $f_{t,a} \in T_\alpha(A)$ such that $f_{t,a}(t) = a$.*
2. *The set $M_{t,\mathcal{M}} = \{f \in T_\alpha(A) \colon f(t) \in \mathcal{M}\}$ is an ideal of $T_\alpha(A)$.*
3. *$M_{t,\mathcal{M}}$ is maximal if and only if $\mathcal{M}$ is maximal.*
4. *M is a fixed maximal ideal of $T_\alpha(A)$ if and only if $M = M_{t_0,\mathcal{M}}$ for some $t_0 \in [0,1]$ and $\mathcal{M}$ a maximal ideal in A.*

Maximal ideals

Theorem 4.

Let A be a unital Banach algebra and let α be an automorphism of A. Then every proper ideal in $T_\alpha(A)$ is fixed.

Maximal ideals

Hence, if A is simple then every maximal ideal in $T_\alpha(A)$ is the kernel of an evaluation.

The isomorphism of the mapping tori maps $\ker \mathrm{ev}_t$ to $\ker \mathrm{ev}_{\lambda(t)}$

$T_\alpha(A)/\ker \mathrm{ev}_0$ is isomorphic to A.

Thanks!

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