Indirect Measurements and State Purification

Miguel Ballesteros. IIMAS (UNAM), Mexico.

Joint work with:
M. Fraas UC Davis
J. Fröhlich (ETH Zurich, Switzerland)
B. Schubnel (SBB, Switzerland)

Main Question:

What information on a quantum system, S, can be extracted from long, time-ordered sequences of direct (projective) measurements ?

Inspired by:

- Experiments of S. Haroche et al.
- Theoretical work by M. Bauer and D. Bernard T. Benoist, C. Pellegrini, H. Maassen and B. Kümmerer.

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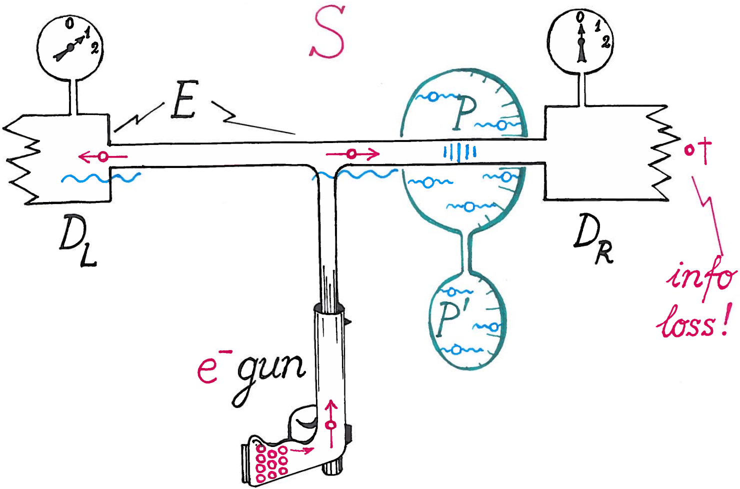
S is the composition of two subsystems, P and E, where P is the system we actually wish to study, while E consists of all the experimental equipment - probes, detectors, and other measuring devices - used to observe.

Example:



General Framework,

Observables at Infinity

and Emergence of Facts

We consider an isolated quantum system S characterized by the data $(\mathcal{H}_S, \{U(t,s)\}_{t,s\in\mathbb{R}}, \mathcal{E})$, where

- $\mathcal{H}_S$ is the Hilbert space of pure state vectors of S,
- $\{U(t,s)\}_{t,s\in\mathbb{R}}$ is a family of unitary operators on $\mathcal{H}_S$ representing time evolution.
- $\mathcal{E} \subset \mathcal{B}(\mathcal{H}_S)$ is a von Neumann -algebra of operators on $\mathcal{H}_S$ representing physical quantities of S that can be measured/observed in projective measurements.

We additionally suppose that there is an **instrument** consisting in a finite family of commuting projections $\{\pi_\xi\}_{\xi\in\sigma}$, such that

$$\sum_{\xi} \pi_\xi = \mathbf{1}_{\mathcal{H}_S}.$$

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- Repeated projective measurements of quantities in $\mathcal{E}$ are carried out at times $i \in \mathbb{N}$.
- The time evolution of the projections π_ξ are given by

$$\pi_\xi(j) := U(0,j)\pi_\xi U(j,0) \in \mathcal{E}, \qquad j \in \mathcal{N}.$$

- For every initial density matrix ρ we define

$$\mu_\rho(\xi_1, ..., \xi_n) := \mathrm{Tr}(\pi_{\xi_n}(n) \cdots \pi_{\xi_1}(1)\rho\pi_{\xi_1}(1) \cdots \pi_{\xi_n}(n)),$$

the probability for observing a sequence of measurement results, $\xi_i \in \sigma$, observed at times i, $i = 1, 2, ..., n$, corresponding to "events" $(\pi_{\xi_1}(1), ..., \pi_{\xi_n}(n))$.

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We denote by $\Xi := \sigma^{\mathbb{N}}$, with elements

$$\underline{\xi} := (\xi_1, \xi_2, \cdots).$$

We define the measure space $(\Xi, \mathcal{F}, \mu_\rho)$, where $\mathcal{F}$ is $\sigma-$algebra generated by cylinder sets and the measure μ_ρ is extended to $\mathcal{F}$ (we use Kolmogorov Theorem).
We denote by $\mathcal{F}_n$ the σ-algebra generated by cylinder sets of the form $\sigma^n \times C \subset \Xi$, where C is a cylinder set. Then we set

$$\mathcal{F}_\infty = \bigcap_n \mathcal{F}_n. \quad (1)$$

This is the algebra of tail-events. Elements of this algebra do not depend on any finite number of coordinates.

Tail Observables: Representation of the algebra of observables at infinity

We denote by

$$\mathcal{E}_{\geq n}$$

the von Neumann algebra generated by projections of the form $\pi_\xi(m)$, or every $m \geq n$ and every $\xi \in \sigma$. It represent all observables (operators) that might be observed after time n. The algebra of tail-observables is

$$\mathcal{E}_\infty := \bigcap_n \mathcal{E}_{\geq n}.$$

For every $\mathcal{F}$-measurable function f, we define an operator $\Phi(f)$ through the duality

$$\mathrm{Tr}(\rho\Phi(f)) = \int f d\mu_\rho. \quad (2)$$

Then Φ defines a POVM (positive operator valued measure), [B-Fraas-Froehlich-Schubnel].
Next we set

$$\mathcal{O}_\infty := L^\infty(\Xi, \mathcal{F}_\infty, \Phi)$$

the algebra of observables at infinity.

For every $f \in \mathcal{L}_\infty$, we define the operator $\Phi_{\geq t_m}(f) \in \mathcal{E}$ by the equation

$$\rho(\Phi_{\geq t_m}(f)) := \int_{\underline{\Xi}} f(\underline{\xi}) d\mu_\rho^{(\geq t_m)}(\underline{\xi}^{(m,\infty)}), \tag{3}$$

for every normal state ρ. The measure $\mu_\rho^{(\geq t_m)}$ is obtained by the quantities

$$\mu_\rho^{(\geq t_m)}(\underline{\xi}^{(m,n)}) := \rho\Big(\Pi_{\underline{\xi}^{(m,n)}} \big(\Pi_{\underline{\xi}^{(m,n)}}\big)^*\Big), \tag{4}$$

for every $n \geq m$ (we recall that f does not depend on the first $m - 1$ coordinates).

Assumption

We assume that $\mathcal{E}_\infty$ is contained in the center of $\mathcal{E}$ and that $\Phi_{\geq m}(f) \in \mathcal{E}_\infty$, for every $f \in \mathcal{O}_\infty$ and every $m \in \mathbb{N}$.

Theorem (B-Fraas-Froehlich-Schubnel)

Φ restricted to $\mathcal{O}_\infty$ is a representation (a $*$-algebra homomorphism).

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Theorem (B-Fraas-Froehlich-Schubnel)

Φ restricted to $\mathcal{O}_\infty$ is a representation (a *-algebra homomorphism).

For every $f \in \mathcal{O}_\infty$ denote by μ_ρ^f the measure

$$\mu_\rho^f(\Delta) := \int_\Delta f d\mu_\rho.$$

For every density matrix ρ define $\Phi_*(f)\rho$ by the duality

$$\mathrm{Tr}\big((\Phi_*(f)\rho)A\big) = \mathrm{Tr}\big(\rho\Phi(f)A\big), \qquad A \in \mathcal{E}.$$

Theorem (B-Fraas-Froehlich-Schubnel)

The following equation holds true:

$$\mu_\rho^f = \mu_{\Phi_*(f)\rho} \tag{5}$$

$\mathcal{F}_\infty$-ergodic desintegration of the measures μ_ρ.

Theorem (B-Fraas-Froehlich-Schubnel)

There exist a probability space $(\Xi_\infty, \Sigma_\infty, P)$ *and a family of measures* $\mu_\nu, \nu \in \Xi_\infty$*, such that*

$$\mu_\rho = \int_{\Xi_\infty} \mu_\nu dP,$$

and the measures $\mu_\nu, \nu \in \Xi_\infty$*, are mutually singular and* $\mathcal{F}_\infty -$*ergodic (when restricted to* $\mathcal{F}_\infty$ *take only the values* 0 *and* 1*).*

The quantities $\nu \in \Xi_\infty$ are interpreted as facts of the system.

A Family of Models

of Quantum Systems

Non-Demolition Measurements

The system S is composed of a subsystem P of interest to an experimentalist and of a subsystem E consisting of a measurement device, so that $\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_E$.
The reduced evolution on P is encoded in a family of completely positive maps $\widetilde{\Phi}_\xi$ $(\xi \in \sigma)$ acting on $\mathcal{B}(\mathcal{H}_P)$, describing the evolution of observables. The evolution of states is given by the dual maps $\widetilde{\Phi}_{*\xi}$. Then we have

$$\widetilde{\mu}_\rho(\xi_1, \cdots, \xi_k) = \mathrm{Tr}(\widetilde{\Phi}_{*\xi_k} \circ \cdots \circ \widetilde{\Phi}_{*\xi_1}[\rho]). \tag{6}$$

The corresponding reduced state $\widetilde{\rho}^{(k)}(\xi_1, \cdots, \xi_k)$ of P is given by

$$\widetilde{\rho}^{(k)}(\xi_1, \cdots, \xi_k) = \frac{\widetilde{\Phi}_{*\xi_k} \circ \cdots \circ \widetilde{\Phi}_{*\xi_1}[\rho]}{\mathrm{Tr}(\widetilde{\Phi}_{*\xi_k} \circ \cdots \circ \widetilde{\Phi}_{*\xi_1}[\rho])}. \tag{7}$$

The non-demolition property is encoded by the condition: $\widetilde{\Phi}_\xi \circ \widetilde{\Phi}_{\xi'} = \widetilde{\Phi}_{\xi'} \circ \widetilde{\Phi}_\xi$, for all $\xi, \xi' \in \sigma$ (that implies the exchangeability of the measures). We denote by $\{\Pi_\nu\}_{\nu \in \Xi_\infty}$ a joint spectral decomposition of the commuting family $\{\widetilde{\Phi}_\xi[1]\}_{\xi \in \sigma}$.

Theorem (Purification)

There exists a random variable $\Theta : \Xi \to \Xi_\infty$ *such that*

$$\left\| \widetilde{\rho}^{(k)} - \frac{\Pi_\Theta \rho \Pi_\Theta}{\mathrm{Tr}(\Pi_\Theta \rho \Pi_\Theta)} \right\| \to 0, \qquad \widetilde{\mu}_\rho\text{- a.s.}, \tag{8}$$

as $k \to \infty$.

Non non-Demolition Measurements:

(Indirect measurements of physical quantities varying slowly in time)

We keep the formalism of the previous section but drop the non-demolition hypothesis. The complete positive maps are now denoted by $\Phi^{(k)}_{\underline{\xi}}$ and only depend on the history $\underline{\xi}^{(k)} = (\xi_1, ..., \xi_k)$ and the time k. We define the density matrices and measures as before, but we drop the tildes.

Assumption

We assume that there exist constants $d_1 \in [0, 1)$ and $d_2 \in (d_1, 1]$ such that, for all $n \in \mathbb{N}$, for every $\xi \in \sigma$ and every $\underline{\xi}$ with $\xi_k = \xi$,

(i) $\|\Phi^{(k)}_{*\underline{\xi}} - \widetilde{\Phi}_{*\xi}\| \leq d_1 \|\widetilde{\Phi}_{*\xi}\|$,

(ii) $\mathrm{Tr}(\widetilde{\Phi}_{*\xi}\rho) \geq d_2 \|\widetilde{\Phi}_{*\xi}\|$, *for all density matrices ρ on $\mathcal{H}_{\overline{P}}$.*

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Theorem (Jump Process)

Let $\varepsilon \in (0,1]$*. If* r *is large enough and if* d_1 *is small enough, then*

$$\mu_\omega\left\{\underline{\xi} \;\middle|\; \exists \nu \in \sigma(\mathcal{N}) \, : \, \|\rho^{(r)}(\underline{\xi}) - \Pi_\nu \rho^{(r)}(\underline{\xi}) \Pi_\nu\| \le \varepsilon\right\} \ge 1 - \varepsilon, \quad (9)$$

uniformly with respect to ρ *and* r*.*

Some Proofs (Ideas)

Φ is a POVM:

Let $f \in L^\infty(\Xi)$. For every positive linear functional ρ, we have that

$$\left|\int_\Xi f(\underline{\xi})\mathrm{d}\mu_\rho(\underline{\xi})\right| \leq \int_\Xi ||f||_\infty \mathrm{d}\mu_\rho(\underline{\xi}) = ||f_\infty||\rho(1).$$

This bound implies that $\Phi(f) \in \mathcal{E}^-$. The properties of a POVM are inherited from the corresponding properties of integrals. For example $\rho(\Phi(\Xi)) = 1$ for all states ρ implies $\Phi(\Xi) = 1$. Next we remind that, by definition,

$$\rho\big(\Phi(\Delta)\big) = \mu_\rho(\Delta), \tag{10}$$

for any positive functional ρ. Therefore, for any disjoint sequence $(\Delta_n)_{n\in\mathbb{N}}$,

$$\rho\Big(\Phi(\bigcup_n \Delta_n)\Big) = \mu_\rho\Big(\bigcup_n \Delta_n\Big) = \sum_n \mu_\rho(\Delta_n) = \sum_n \rho(\Phi(\Delta_n)). \tag{11}$$

This shows that

$$\Phi(\bigcup_n \Delta_n) = \sum_n \Phi(\Delta_n), \tag{12}$$

where the series converges in the σ-weak topology (i.e. it also converges weakly).

Φ is a *homomorphism, restricted to $\mathcal{O}_\infty$:

Lemma

For every function $f \in L^\infty(\Xi)$ there exists an increasing sequence j_n and a sequence of functions $f_n \in L^\infty(\Xi)$, only depending only on the first j_n coordinates, such that

$$w - \lim_{n\to\infty} \Phi_{\geq m}(|f_n - f|) = 0.$$

If in addition $f \in \mathcal{O}_\infty$, then f_n can be chosen such that it only depends on the coordinates $i_n, \cdots, j_n$, for some increasing sequences $(i_n)_{n\in\mathbb{N}}$, $(j_n)_{n\in\mathbb{N}}$ with $m \leq i_n < j_n$.

An explicit calculation shows that

$$\Phi(f_n) = \sum_{\underline{\xi}^{(m-1)}} \Pi_{\underline{\xi}^{(m-1)}} \Phi_{\geq t_m}(f_n) \left(\Pi_{\underline{\xi}^{(1,m-1)}}\right)^*. \qquad (13)$$

Taking the limit when n tends to ∞, using our assumptions, we get

$$\Phi(f) = \sum_{\underline{\xi}^{(m-1)}} \Pi_{\underline{\xi}^{(m-1)}} \Phi_{\geq t_m}(f_n) \left(\Pi_{\underline{\xi}^{(1,m-1)}}\right)^* = \Phi(f_n) \qquad (14)$$

Next, we show that $\Phi|_{\mathcal{O}_\infty}$ is a *homomorphism from $\mathcal{O}_\infty$ to $\mathcal{E}_\infty$. It suffices to show that $\Phi(f \cdot \chi_\Delta) = \Phi(f) \cdot \Phi(\chi_\Delta)$ for any cylinder set $\Delta \in \Sigma$. The previous equality then easily extends to arbitrary functions $f, g \in \mathcal{O}_\infty$ by a density argument. (Moreover, compatibility with the * operation is obvious).

Let f_n be approximations of the function f as above and assume that $\Delta \in \Sigma_{1,m-1}$ for some $m \in \mathbb{N}$. If n is such that $i_n > m$, we have that

$$\Phi(f_n \cdot \chi_\Delta) = \sum_{\underline{\xi}^{(m-1)}} \chi_\Delta(\underline{\xi}^{(m-1)}) \, \Pi_{\xi_1} \ldots \Pi_{\xi_{m-1}} \tag{15}$$

$$\Big(\sum_{\underline{\xi}^{(m,j_n)}} f_n(\underline{\xi}^{(i_n,j_n)}) \Pi_{\xi_m} \ldots \Pi_{\xi_{j_n}} \Pi_{\xi_{j_n}} \ldots \Pi_{\xi_{i_m}} \Big) \Pi_{\xi_{m-1}} \ldots \Pi_{\xi_1} .$$

Taking the limit when n tends to infinity we get

$$\Phi(f \cdot \chi_\Delta) = \sum_{\underline{\xi}^{(m-1)}} \chi_\Delta(\underline{\xi}^{(m-1)})\, \Pi_{\xi_1} \ldots \Pi_{\xi_{m-1}} \Phi_{\geq t_m}(f) \Pi_{\xi_{m-1}} \ldots \Pi_{\xi_1} = \Phi(\chi_\Delta)\Phi(f), \tag{16}$$

where we use our assumptions and (14).

State Associated with $f \in \mathcal{O}_\infty$:

(Proof of Eq. (5))

Let $f \in \mathcal{O}_\infty$. For a characteristic function χ_Δ of a cylinder set Δ, we have, using our assumptions, that

$$\mu_{\Phi(f)^*\rho}(\Delta) = \rho(\Phi(f)\Phi(\chi_\Delta)).$$

Eq. (16), we have established that $\Phi(f)\Phi(\chi_\Delta) = \Phi(f\chi_\Delta)$. Hence $\mu_{\Phi(f)^*\rho}(\Delta) = \mu^f_\rho(\Delta)$.