Universality of Free Random Variables: Atoms for Non Commutative Rational Functions

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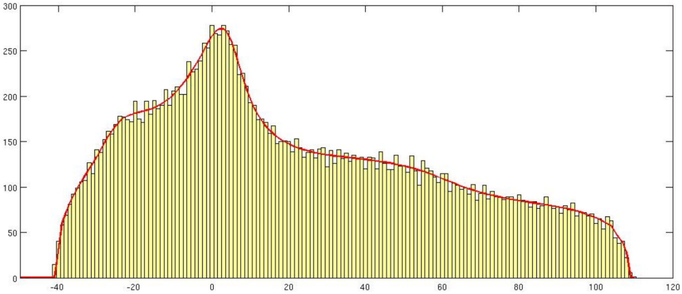
Asymptotic Freeness



Polynomial (from BSVT 2015, BMS 2016)

The question of this talk: atoms in polynomials in free variables

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Atom: λ such that $\mu(\lambda) > 0$

Motivation

Let A_n and B_n be selfadjoint matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ y $\rho_1, \ldots, \rho_n$.

- What is the nullity (dim(ker)) $P(A_n, B_n)$?

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- What is the nullity (dim(ker)) $P(A_n, B_n)$?

With more information than the spectrum we cannot answer the question exactly . However we can say something about the next two related questions:

1. What is the minimum nullity (over all matrices A_n and B_n) of $P(A_n, B_n)$?
2. ¿What is the nullity $P(A_n, B_n)$ if A_n are B_n randomly rotated?

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Obvious observation : By shifting $P(A_n, B_n) - \lambda$ we can consider the size of the subespace asociado to λ.

Problem

- $\{A_n\}$ and $\{B_n\}$ two sequences of self-adjoint matrices (deterministic) matrices with limit in distribution, μ and ν.
- U_n a unitary random matrix (with Haar measure on $U(n)$).
- P a non commutative polynomial.

Let μ_P be the limit distribution of $P(A_n, U_n B_n U_n^*)$.

¿Can we determine the atomic part of μ_P ?

That is, for each $\lambda \in \mathbb{R}$. ¿What is $\mu_P\{\lambda\}$?

In terms of free probability

Let $X_1, X_2, \cdots X_n$ be free random variables.

¿For a given polynomial, we can determine the atoms of$P(X_1, ..., X_d)$ **?**

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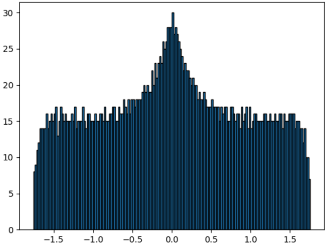
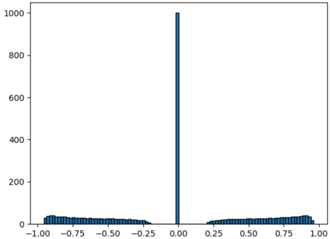
From which information of X_i can we obtain the atoms of $P(X_1, ..., X_d)$?

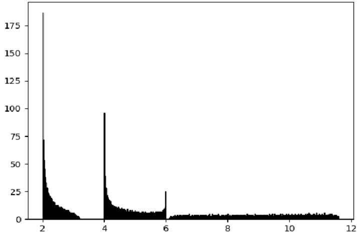


$$\mu_{D_1} = \mu_{D_2} = \frac{1}{3}(\delta_1 + \delta_2 + \delta_3)$$



$$\mu_{D_1} = \frac{2}{3}\delta_1 + \frac{1}{3}\delta_2, \mu_{D_2} = \frac{1}{3}(\delta_1 + \delta_2 + \delta_3).$$

$$i(XY - YX) = i(D_1 U D_2 U - U D_2 U D_1)$$



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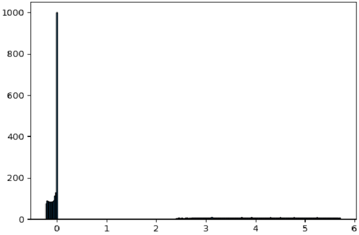

$\mu_{D_1} = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_1, \mu_{D_2} = \frac{1}{3}(\delta_1 + \delta_2 + \delta_3).$

Anticommutator $XY + YX = D_1 U D_2 U + U D_2 U D_1$

Previous results

- If X_i have no atoms, then also $P(X_1, ..., X_d)$ no atoms. (Mai-Speicher-Weber 15, Charlesworth-Shlyakhtenko 15) .
- Exact solution for $X_1 + X_2$ (Bercovici-Voiculescu 98) and $X_1 X_2$ Belinschi (2003).
- If the atoms X_i all have rational sizes, also the ones of $P(X_1, ...X_d)$ Shlyaktenko-Skoufranis (2015).

Our main theorem: Minimality of the free case

Theorem

Let $X_1, \ldots, X_d$ and $Y_1, \ldots, Y_d$ be normal variables in a tracial W^*-probability spaces with $X_1, \ldots, X_d$ being $*$-free and such that, for all $k = 1, \ldots, d$ and each $\lambda \in \mathbb{C}$, we have

$$\mu_{X_i}(\{\lambda\}) = \mu_{Y_i}(\{\lambda\}).$$

Then, for each selfadjoint polynomial P in d non-commuting variables in d non-commuting variables,

$$\mu_{P(X)}(\{\lambda\}) \leq \mu_{P(Y)}(\{\lambda\}).$$

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- P can be a rational function or matrix.

Atoms in Free Random Variables

Theorem (A. Cebron, Speicher, Yin, 2021+)

Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ two d-tuples of free random variables such that, for each $1 \leq i \leq d$ y $\lambda \in \mathbb{C}$, we have

$$\mu_{X_i}(\{\lambda\}) = \mu_{Y_i}(\{\lambda\}).$$

Then, for each polynomial P, and each $\lambda \in \mathbb{C}$, we have

$$\mu_{P(X)}(\{\lambda\}) = \mu_{P(Y)}(\{\lambda\}).$$

In other words, the atoms of polynomials in free variables only depend on the atomic part of each of the variables.

Comparison with matrices

Let $X = (X_1, \dots, X_d)$ be a tuple of $*$-free normal random variables. We define

$$\mathcal{X}_m := \{Y = (Y_1, \dots, Y_d) \in M_m(\mathbb{C})^d : \mu^p_{X_k} \leq \mu^p_{Y_k}\}$$

and $\mathcal{X} := \coprod_{m=1}^{\infty} \mathcal{X}_m$.
By our main theorem

$$\mu^p_{P(X)} \leq \inf_{Y \in \mathcal{X}} \mu^p_{P(Y)}.$$

Proposition

Let $X = (X_1, \dots, X_d)$ be a tuple of $*$-free normal random variables. Then for any $P \in \mathbb{C}\langle x_1, \dots, x_d\rangle$

$$\mu^p_{P(X)} = \inf_{Y \in \mathcal{X}} \mu^p_{P(Y)}.$$

Our aim is to use the above theorem to obtain as much information of the atoms for the **free case** as we can from **specific choices** for Y_i's.

Example: Free additive convolution

Theorem (Bervocivi Voiculescu 98)

$\mu \boxplus \nu$ has an atom at $a \in \mathbb{R}$ if and only if there exist $\lambda, \rho \in \mathbb{R}$ such that $\rho + \lambda = a$ and $\mu(\lambda) + \nu(\rho) > 1$. Moreover, if $\mu(\lambda) + \nu(\rho) > 1$, we have $\mu \boxplus \nu(a) = \mu(\lambda) + \nu(\rho) - 1$.

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Lower bound: Simple and intuitive. Take matrices $X, Y \in M_n$.
If $dim(Ker(X - \lambda I)) = m = nt$ and $dim(Ker(Y - \rho I)) = l = ns$, then

$$\begin{aligned} dim(Ker(X + Y - \lambda I - \rho I)) &\geq dim(Ker(X - \lambda I) \cap Ker(Y - \rho I)) \\ &\geq (m + l - n) = n(t + s - 1) \end{aligned}$$

Upper bound: Case 1. $t+s>1$. Consider the matrices,

$$X_n = \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \lambda_{i+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}, Y_n = \begin{pmatrix} \rho_1 & & & & & \\ & \ddots & & & & \\ & & \rho_j & & & \\ & & & \rho & & \\ & & & & \ddots & \\ & & & & & \rho \end{pmatrix}$$

where $i = sn \geq j = n - tn$.
Then

$$X_n + Y_n = \begin{pmatrix} \lambda+\rho_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda+\rho_j & & & & & & \\ & & & \color{red}{\lambda+\rho} & & & & & \\ & & & & \color{red}{\ddots} & & & & \\ & & & & & \color{red}{\lambda+\rho} & & & \\ & & & & & & \lambda_{i+1}+\rho & & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_n+\rho \end{pmatrix}$$

Example: Free additive convolution

$$X_n + Y_n = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} \lambda + \rho_1 & & \\ & \ddots & \\ & & \lambda + \rho_i \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \lambda + \rho & & \\ & \ddots & \\ & & \lambda + \rho \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} \lambda_{j+1} + \rho & & \\ & \ddots & \\ & & \lambda_n + \rho \end{pmatrix}.$$

We see that the size of the eigenspace associated to a is $\dim(A_2) = i - j = (s + t - 1)n$.

Upper bound: Case 1. $t + s < 1$.
We claim that we can reorder the eigenvalues in such a way that $\lambda_i + \rho_i \neq a$, for all i. Taking the matrices $X_n = diag(\lambda_1, \ldots, \lambda_n)$ and $Y_n = (\rho_1, \ldots, \rho_n)$, we see that $X_n + Y_n = diag(\lambda_1 + \rho_1, \ldots, \lambda_n + \rho_n)$ which has no eigenvalue equal to a, as desired.

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Proof of claim: We prove this by induction on n. For $n = 1$ and $n = 2$, it is clear. Consider a reordering of $\{\lambda_i, \rho_i\}$ such $\lambda_i + \rho_i \neq a$ for $i \leq n - 1$ which is possible by induction. Now consider $\lambda_n + \rho_n$. If $\lambda_n + \rho_n \neq a$ we are done. Otherwise, if $\lambda_n + \rho_n = a$, then let

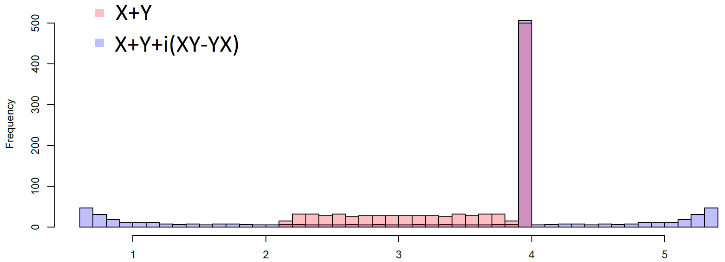
$$S = \{j \in [n-1] \mid \text{such that } \lambda_j = \lambda_n \text{ or } \rho_j = \rho_n\}.$$

If $|S| = n - 1$, by choosing for each j, ρ_j or λ_j, together with λ_n, ρ_n we have that $sn + tn \geq n + 1$, which yields a contradiction. Finally, if $|S| \leq n - 2$, there exists j such that $\lambda_j \neq \lambda_n$ and $\rho_j \neq \rho_n$. Since $\lambda_j + \rho_n \neq a$ and $\lambda_n + \rho_j \neq a$, we get the desired reordering.

(Injective) Polynomials in two variables

Theorem

Let X and Y be free and $a \in \mathbb{R}$. Let P satisfy that for all λ, ρ such that $P(\lambda, \rho) = a$ then $P(\lambda, \tilde{\rho}) \neq a$ and $P(\tilde{\lambda}, \rho) \neq a$ whenever $\lambda \neq \tilde{\lambda}$ and $\rho \neq \tilde{\rho}$.
Then $P(X, Y)$ has an atom at a if and only there are λ and ρ such that X has an atom at λ of size s and Y has an atom at ρ of size t, such that $r = t + s - 1 > 0$ and $P(\lambda, \rho) = a$.
Furthermore, if $r > 0$, and $s(a)$ and $t(a)$ denote the mass of this (unique) atoms, then the mass at a is given by $s(a) + t(a) - 1$.



Unavoidable atoms

Proposition

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider a tuple $X = (X_1, ..., X_d)$ of selfadjoint operators in $\mathcal{M}$. Suppose that for each i, $\mu_i(\lambda_i) \geq t_i$ for some $\lambda_i \in \mathbb{R}$ and $t_i \in [0,1]$. If $t_1 + \cdots + t_d > d - 1$ then for any selfadjoint polynomial $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$, the measure $\mu_{P(X)}$ has an atom at $P(\lambda)$ of size at least $t_1 + \cdots + t_d - (d-1)$, where $\lambda := (\lambda_1, \ldots, \lambda_d)$.

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Proof

- For each i, let p_i be such $Xp_i = \lambda_i p_i$ for all $i \in \{1, \ldots, d\}$ and $\tau(p_i) = t_i$.
- Then for $p := \min(p_1, \ldots, p_d)$, we have $P(X)p = P(\lambda)p$.
- Finally $\tau(p) \geq \tau(p_1) + \cdots + \tau(p_d) - (d-1) = t_1 + \cdots + t_d + d - 1$.

Injective Polynomials

Definition

Let $P \in \mathbb{C}\langle x_1, \dots, x_d \rangle$ be given. For $a \in \mathbb{R}$, if $P(\rho_1, \dots, \rho_i, \dots, \rho_d) = a$ and

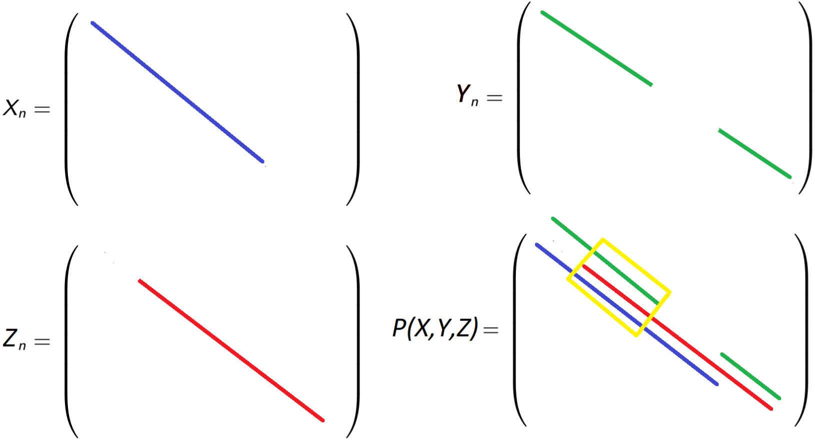
$$P(\rho_1, \dots, \rho_i, \dots, \rho_d) \neq P(\rho_1, \dots, \tilde{\rho}_i, \dots, \rho_d)$$

whenever $\rho_i \neq \tilde{\rho}_i$ for any given scalar values $\rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_d$, then we say P is *injective* for a.

Theorem

Let $P \in \mathbb{C}\langle x_1, \dots, x_d \rangle$ be injective for $a \in \mathbb{R}$ and let $a = P(\lambda_1, \dots, \lambda_d)$. Suppose that $X_1, \dots, X_d$ are free selfadjoint random variables and $X_i \sim \mu_i$ with $\mu_i(\{\lambda_i\}) = t_i$. If $t_1 + \ldots + t_d \geq d - 1$ then the distribution of $P(X_1, \dots, X_d)$ has an atom at a of size exactly $t_1 + \ldots + t_d - (d-1)$.

Idea of proof for 3 variables.



Not every polynomial is injective for all atoms

Theorem (Belinschi 2003)

If $\mu \in \mathcal{P}(\mathbb{R}^+)$ then $\mu \boxtimes \nu$ has an atom at $a \in \mathbb{R} \setminus \{0\}$ if and only if there exist $\lambda, \rho \in \mathbb{R}$ such that $\rho\lambda = a$ and $\mu(\lambda) + \nu(\rho) > 1$. Moreover

- *If $\mu(\lambda) + \nu(\rho) > 1$, we have $\mu \boxtimes \nu(a) = \mu(\lambda) + \nu(\rho) - 1$.*
- *The atom at 0 is given by $\mu \boxtimes \nu\{0\} = \max(\mu\{0\}, \nu\{0\})$.*

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Lower bound follows from directly (next page for general case).

Unavoidable atoms

Proposition

Let $X, Y_1, ..., Y_d$ be selfadjoint operators in some tracial W^*-probability space $(\mathcal{M}, \tau)$. Suppose that $\mu_X(\{0\}) \geq t$. Then for any selfadjoint polynomial P of the form

$$P(x, y_1, ..., y_d) = \sum_{i=1}^{k} Q_{i,1}(x, y_1, ..., y_d) x Q_{i,2}(x, y_1, ..., y_d),$$

the analytic distribution of $P(X, Y_1, ..., Y_d)$ has an atom at 0 whose size is at least $\max(kt - (k-1), 0)$.

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Proof

$$\begin{aligned} rank(P(X, Y_1, \ldots, Y_d)) &\leq \sum_{i=1}^{k} rank(Q_{i,1}(X, Y_1, ..., Y_d) X Q_{i,2}(X, Y_1, ..., Y_d)) \\ &\leq k rank(X) = k(1-t), \end{aligned}$$

$\mu \boxtimes \nu\{0\} \leq \max(\mu\{0\}, \nu\{0\})$. We may assume that $\mu \sim X_n^2$ and $\nu \sim Y_n$. Consider X_n and Y_n,

$$X_n = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, Y_n = \begin{pmatrix} \rho_1 & & & & & \\ & \ddots & & & & \\ & & \rho_l & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

with $\lambda_i \neq 0$ for $0 \leq i \leq m$ and $\rho_j \neq 0$, for $0 \leq j \leq l$. Then, if $r = min(l, m)$, we have

$$X_n Y_n X_n = \begin{pmatrix} \lambda_1\rho_1\lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r\rho_r\lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

The result follows since $Null(X_n Y_n X_n) = max\{Null(X_n), Null(X_n)\}$.

Theorem

Let $X_1 \ldots, X_d$, be free. The possible atoms of $P(X_1, ..., X_d)$ are contained in the set

$$\{P(\rho_1, ..., \rho_d) \mid \rho_i \text{ is atom of } X_i, \ \forall i = 1, \ldots, d\}.$$

Furthermore, if $X_1, \ldots, X_d \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{R}$, then $\mu_{P(X_1,\ldots,X_d)}(\{\lambda\}) \leq k(\lambda)/n$, where

$$k(\lambda) := \min_{\sigma_1,\ldots,\sigma_d \in S_n} |\{j \in \{1, ...n\} : p(\lambda^{(1)}_{\sigma_1(j)}, ..., \lambda^{(d)}_{\sigma_d(j)}) = \lambda\}|,$$

where, for $i = 1, \ldots, d$, $\{\lambda^{(i)}_j\}_{j=1}^n$ denotes the eigenvalues of X_i.

Commutative Matrices: Further consequences

Theorem

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where, for $i = 1, \ldots, d$, $\{\lambda^{(i)}_j\}_{j=1}^n$ denotes the eigenvalues of X_i.
Example: $i(XY - YX)$ can only have atoms at 0. We'll see $k(0)$ is not an optimal bound.

2x2 matrices

Now we consider $X_n, Y_n \in M_{2n}(\mathbb{C})$ matrices consisting of diagonal blocks of size 2×2, $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ on their diagonals, respectively.

$$X_n = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, Y_n = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}.$$

Similar as before

$$p(X_n, Y_n) = \begin{pmatrix} p(A_1, B_1) & 0 & \cdots & 0 \\ 0 & p(A_2, B_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p(A_n, B_n) \end{pmatrix}.$$

How to choose A_i and B_i?

Example: Commutator

Theorem

Let X and Y be free random variables. Let t and s be the size of the largest atom of X and Y, respectively, i.e.,

$$t = \max\{\mu_X(\{a\}) \mid a \in \mathbb{R}\} \qquad \textit{and} \qquad s = \max\{\mu_Y(\{b\}) \mid b \in \mathbb{R}\}.$$

Then

1. $i(XY - YX)$ *has an atom at* 0 *of size given by* $max(2t-1, 2s-1, 0)$.
2. $i(XY - YX)$ *has no futher atoms.*

Case $t, s \leq 1/2$**.** Our aim is to show that $i(XY - YX)$ has no atom at 0. We take $X_n, Y_n \in M_{2n}$ and reorder the eigenvalues so that $\lambda_{2i-1} \neq \lambda_{2i}$ and $\rho_{2i-1} \neq \rho_{2i}$

Now, consider the block-diagonal matrices

$$X_n = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}, \qquad Y_n = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}$$

with

$$A_i = \begin{pmatrix} \lambda_{2i-1} & 0 \\ 0 & \lambda_{2i} \end{pmatrix} \qquad \text{and} \qquad B_i = \frac{1}{2} \begin{pmatrix} \rho_{2i-1} + \rho_{2i} & \rho_{2i-1} - \rho_{2i} \\ \rho_{2i-1} - \rho_{2i} & \rho_{2i-1} + \rho_{2i} \end{pmatrix}.$$

So

$$[X_n, Y_n] = \begin{pmatrix} [A_1, B_1] & & \\ & \ddots & \\ & & [A_n, B_n] \end{pmatrix}$$

and it is enough to prove that $[A_i, B_i]$ is invertible. But

$$[A_i, B_i] = \frac{1}{2} \begin{pmatrix} 0 & (\lambda_{2i} - \lambda_{2i-1})(\rho_{2i} - \rho_{2i-1}) \\ -(\lambda_{2i} - \lambda_{2i-1})(\rho_{2i} - \rho_{2i-1}) & 0 \end{pmatrix}.$$

whose determinant is $(\lambda_{2i} - \lambda_{2i-1})^2(\rho_{2i} - \rho_{2i-1})^2 \neq 0$.

Anticommutator

Theorem

Let X and Y be free random variables and let $Z = XY + YX$.
i)The size of the atom at 0 of Z is given by

$$l := \max\{2t-1, 2s-1, s+u-1, t+r-1, 0\},$$

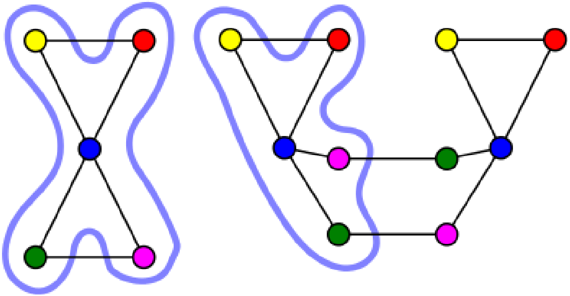
where

1. *t is the size of the atom at 0 of X;*
2. *s is the size of the atom at 0 of Y;*
3. *u is the size of the largest atom outside of 0 of X;*
4. *r is the size of the largest atom outside of 0 of Y.*

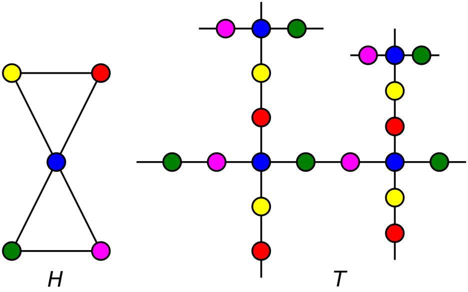
ii) For any $a \neq 0$, Z has an atom at a if and only if there exist weights $s(a)$ and $t(a)$ such that $t(a) + s(a) - 1 > 0$, X has an atom at λ of size $s(a)$ and Y has an atom at ρ of size $t(a)$ and $2\lambda\rho = a$.
The size of the atom of Z at $a \neq 0$ is given by $\max\{s(a) + t(a) - 1, 0\}$.

An Application: Spectrum of Universal Covering of Graphs

- Let $G = (V, E)$ be a finite graph.
- A covering graph of G is a graph $H = (W, F)$ such that there is a function $f : W \to V$ which is a local isomorfism
- **Local isomorfism** the neighborhood of v (v y sus vecinos junto con las conexiones que van a v) in H is sent bijectively to the neighborhood of $f(v)$ en G.



Universal Covering of Graphs



Spectrum

- We want to relate the spectrum of G to that of $T(G)$.
- For G we take the eigenvalues of $A(G)$.
- For $T(G)$ we consider the *density of states* , $\mu_{T(G)}$: For each v we choose a preimage of v, which we call $f^{-1}(v)$. Then

$$\mu_{T(G)} = \sum_{v \in V} \mu_{f^{-1}(v)}$$

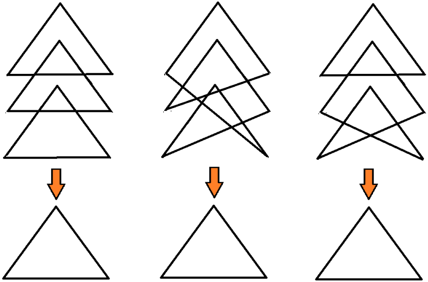
Theorem (Banks, Garza-Vargas, Mukherjee 2020)

The point spectrum of $T(G)$is contained in the set of eigenvalues of $A(G)$.

Why free Probability?

Theorem

For each k, let $P_k(G)$ be a random covering of size k of G, and let μ_k be its average eigenvalue distribution. Then $\mu_k \to \mu_{T(G)}$



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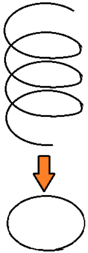

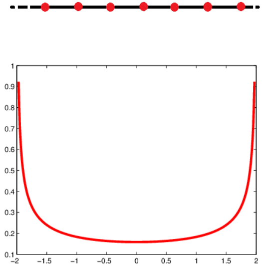

Why free Probability?

Theorem

For each k, let $P_k(G)$ be a random covering of size k of G, and let μ_k be its average eigenvalue distribution. Then $\mu_k \to \mu_{T(G)}$

We can make a k- random covering changing each 1 in A_{ij} of G by a random permutationof size k and its inverse in A_{ji}.

$$A(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & P_{\sigma_1} & P_{\sigma_2} \\ P_{\sigma_1}^{-1} & 0 & P_{\sigma_3} \\ P_{\sigma_2}^{-1} & P_{\sigma_3}^{-1} & 0 \end{pmatrix}$$

Asymptotic freeness: $P_{\sigma_1} \to u_i$. u_i are free Haar unitaries.

Why free Probability?

Theorem

(Bordenave, Collins 2019) If G has adjacency matrix $A=(A_{i,j})_{i,j}$ y $\{u_{i,j}\}_{0\leq i,j\leq n}$is a family of Haar unitaries such that $u_{i,j}=u_{j,i}^{-1}$ and $\{u_{i,j}\}_{i>j}$ are free. Let $U(A)$, be defined by

$$U(A)_{i,j}=A_{i,j}u_{i,j}.$$

The density of states $T(G)$ coincides with the distribution of $U(G)$.

$$A(G)=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & u_1 & u_2 \\ u_1^{-1} & 0 & u_3 \\ u_2^{-1} & u_3^{-1} & 0 \end{pmatrix}$$

¿ Why free Probability?

Theorem

(Bordenave, Collins 2019) The density of states $T(G)$ coincides with the distribution of $U(G)$.

Theorem (Banks, Garza-Vargas, Mukherjee 2020)

The pint spectrum of $T(G)$ is contained in the spectrum of $A(G)$. Moreover, if $\lambda \in \mathbb{R}$, $\mu_{T(G)}(\lambda) \leq \mu_{A(G)}(\lambda)$.

Theorem (A. Cebron, Yin, Speicher 2021)

Let $Y = (Y_1, \dots, Y_d)$ be operatores and let $X = (X_1, \dots, X_d)$ be a a fmily of free X_i such that

$$\mu^p_{X_k} \leq \mu^p_{Y_k}.$$

If $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, then $\mu_{A(Y)}(\lambda) \leq \mu_{A(X)}(\lambda)$.

Thanks.