Gaussian states: analytical and combinatorial approaches

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Mini-Meeting-IIMAS-2023

IIMAS-UNAM, CDMX

November, 2023

[1] J. R. Bolaños-Servín, R. Quezada and J. I. Rios-Cangas, *Weyl Moments and Quantum Gaussian States*, Reports on Mathematical Physics, 2022.

Outline:

1. Analytic approach to Gaussian states:
    - one-mode representation of the CCR's,
    - analytic Gaussian states, mean value vector and covariance.
2. Combinatorial approach to Gaussian states:
    - moments of the field operator on Gaussian states (Weyl moments),
    - Gaussian maps on algebras,
    - boson Gaussian states.
3. Equivalence of analytic and combinatorial approaches.
4. Examples of Gaussian states and channels:
    - Gibbs state,
    - coherent states and channels,
    - squeezed states and channels.

On h $= L_2(\mathbb{R}, \mu)$ with $d\mu = \mathbb{1}(x)dx$ and $\mathbb{1}(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$.
Consider

$$Qf(x) = xf(x)\,, \qquad \text{position}$$

$$Pf(x) = -i\frac{d}{dx}f(x)\,, \qquad \text{momentum}$$

Also,

$$\text{annihilation} \qquad a^\dagger := \frac{1}{\sqrt{2}}(Q - iP)\,,$$

$$\text{creation} \qquad a := \frac{1}{\sqrt{2}}(Q + iP)\,.$$

$$Q = Q^*\,, \quad P = P^*\,, \quad a^* = a^\dagger\,, \quad a^{\dagger *} = a\,.$$

CCR's

$$[Q, P] = iI \qquad \text{and} \qquad [a, a^\dagger] = I$$

Weyl operator

$$W_z = e^{za^\dagger - \bar{z}a} = e^{-2ip(z)} \quad (z \in \mathbb{C})$$

donde $p(z) := -\frac{1}{2i}(za^\dagger - \bar{z}a)$ is the field operador.

- $p(z)$ is self-adjoint.
- $p(tz_1 + z_2) = t\,p(z_1) + p(z_2),\ t \in \mathbb{R}$
- W_z is a unitary operator, with $W_0 = I$.
- $W_z^* = W_{-z}$.
- $W_z W_u = e^{-i(z,u)} W_{z+u}$.
- $W_{tz} W_{rz} = W_{(t+r)z}, \qquad t, r > 0$.
- $t \mapsto W_{tz}$ is a strongly continuous unitary group.

The *Wigner transform* or quantum Fourier transform is

$$\mathcal{F}[\rho](z) := \frac{1}{\sqrt{\pi}}\mathrm{tr}\,(\rho W_z) = \frac{1}{\sqrt{\pi}}\mathrm{tr}\left(\rho e^{-2ip(z)}\right), \quad \rho \in L_1\,(\mathrm{h})\ ; \quad z \in \mathbb{C}\,.$$

We consider the symplectic space $(\mathbb{C}, (\cdot,\cdot))$, with $(z,u) := \mathrm{Re}\,\bar{z}u$; $\ z,u \in \mathbb{C}$.

Gaussian state

A state $\rho \in L_1(\mathrm{h})$ is *Gaussian* if there exist $w \in \mathbb{C}$ and $S \in \mathcal{B}_{\mathbb{R}}(\mathbb{C})$ real symmetric such that

$$\mathcal{F}[\rho](z) = \frac{1}{\sqrt{\pi}} e^{-i(w,z)-\frac{1}{2}(z,Sz)}, \quad \forall z \in \mathbb{C}\,.$$

- If $w = \sqrt{2}(l - im)$ we call l, m the momentum and mean position vectors.
- S is the covariance matrix.

$$S = \begin{pmatrix} (1,S1) & (1,Si) \\ (i,S1) & (i,Si) \end{pmatrix},$$

Moments of an observable in a state

We will show that, under suitable conditions on the pair (A, ρ), all moments of an observable A in a state ρ can be computed by using derivatives of the functions $\mathrm{tr}\left(\rho e^{-t(iA)^n}\right)$, which involve $\mathrm{tr}\left(\rho A^n\right)$, $n \in \mathbb{N}$.

- Formally the n-th moment of A in a state ρ is $\mathrm{tr}\left(\rho A^n\right)$.
- But $\mathrm{tr}\left(\rho A^n\right)$ requires a rigorous definition if the observable A is unbounded.
- $\mathrm{tr}\left(\rho A^n\right)$ exists for all $n \geq 1$ if the pair (ρ, A) is regular enough: for instance if the observable A is bounded and ρ is any state.
- If A is an unbounded observable, $u \in \mathrm{dom} A$ and $\rho = |u\rangle\langle u|$, then $\mathrm{tr}\left(\rho A\right) = \langle u, Au\rangle$

Yosida approximations

To define moments of an unbounded observable in a state ρ, we use the well know properties of Yosida's approximations, for unbounded selfadjoint operators.
Given $\epsilon > 0$ and an infinitesimal generator Λ of a strongly continuous semigroup of contractions $e^{-t\Lambda}$, with $t \geq 0$, it follows that

- $(I+\epsilon\Lambda)^{-1}$ is a contraction in $\mathcal{B}(\mathcal{H})$, and $\Lambda_\epsilon := \Lambda(I+\epsilon\Lambda)^{-1} \in \mathcal{B}(\mathcal{H})$.
- In addition, for $u \in \mathcal{H}$,

$$\lim_{\epsilon\to 0}(I+\epsilon\Lambda)^{-1}u = u\,, \quad \lim_{\epsilon\to 0} e^{-t\Lambda_\epsilon}u = e^{-t\Lambda}u\,,$$

- while for $u \in \mathrm{dom}\Lambda$,

$$(I+\epsilon\Lambda)^{-1}\Lambda u = \Lambda(I+\epsilon\Lambda)^{-1}u\,, \quad \lim_{\epsilon\to 0}\Lambda_\epsilon u = \Lambda u\,.$$

- The operator Λ_ϵ is the infinitesimal generator of the uniformly continuous semigroup of contractions $e^{-t\Lambda_\epsilon}$.

The above conditions are satisfied if :

- $\Lambda = \pm iA$, where A is an observable, i.e., a selfadjoint operator.
- Λ is any positive self-adjoint operator A (generator of a strongly continuous semigroup of contractions).
- In the last case, $(I + \epsilon A)^{-1}$ and A_ϵ increases to I and A, respectively.

Definition

A non-necessarily bounded selfadjoint operator A is integrable with respect to a state ρ (ρ-integrable for short) if both limits

$$\lim_{\epsilon\to 0} \mathrm{tr}\left(\rho(A_+)_\epsilon\right) \quad \text{and} \quad \lim_{\epsilon\to 0} \mathrm{tr}\left(\rho(A_-)_\epsilon\right)$$

exist. In such a case we write

$$\mathrm{tr}\left(\rho A\right) := \lim_{\epsilon\to 0} \mathrm{tr}\left(\rho[(A_+)_\epsilon - (A_-)_\epsilon]\right) . \tag{1}$$

The usual trace is recovered when A is bounded.

Example: *In the case when* $\rho = |u\rangle\langle u|$ *and* $A = A^*$ *with* $u \in \mathrm{dom\,A}$*, then* A *is* ρ*-integrable and*

$$\mathrm{tr}\,(\rho A) = \langle u, Au \rangle \ .$$

Proof: Since u belongs to the domain of A, it follows that

$$\mathrm{tr}\,(\rho A_+) = \lim_{\epsilon \to 0} \mathrm{tr}\,(\rho (A_+)_\epsilon) = \lim_{\epsilon \to 0} \langle u, (A_+)_\epsilon u \rangle = \langle u, A_+ u \rangle \ .$$

Analogously, $\mathrm{tr}\,(\rho A_-) = \langle u, A_- u \rangle$. In this fashion, one obtains that

$$\mathrm{tr}\,(\rho A) = \lim_{\epsilon \to 0} (\mathrm{tr}\,(\rho (A_+)_\epsilon) - \mathrm{tr}\,(\rho (A_-)_\epsilon)) = \langle u, A_+ u \rangle - \langle u, A_- u \rangle = \langle u, Au \rangle \ ,$$

as required.

The above definition can be extended for any normal operator A as follows

$$\begin{aligned} \operatorname{tr}(\rho A) = & \lim_{\epsilon \to 0} \operatorname{tr}\left(\rho\left(\mathrm{Re}\,\mathrm{A}\right)_{+}\right)_{\epsilon}\Big) - \lim_{\epsilon \to 0} \operatorname{tr}\left(\rho\left(\left(\mathrm{Re}\,\mathrm{A}\right)_{-}\right)_{\epsilon}\right) \\ & + i \operatorname{tr}\left(\rho\left(\left(\mathrm{Im}\,\mathrm{A}\right)_{+}\right)_{\epsilon}\right) - i \lim_{\epsilon \to 0} \operatorname{tr}\left(\rho\left(\left(\mathrm{Im}\,\mathrm{A}\right)_{-}\right)_{\epsilon}\right) , \end{aligned}$$

whenever the limit exists.

“Ampliation" of positive operators and semigroups

For any state ρ and any bounded positive selfadjoint operator A, we have that

$$\operatorname{tr}(\rho A) = \operatorname{tr}\left(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}}\right) = \left\langle A^{\frac{1}{2}}\rho^{\frac{1}{2}}, A^{\frac{1}{2}}\rho^{\frac{1}{2}}\right\rangle_2 = \left\langle \rho^{\frac{1}{2}}, A\rho^{\frac{1}{2}}\right\rangle_2 .$$

In this fashion, one can represent states by unit vectors $\rho^{\frac{1}{2}}$ in the Hilbert space $L_2(\mathrm{h})$. Moreover, one can carry the observables and semigroups defined on h to corresponding observables and semigroups acting on the space $L_2(\mathrm{h})$. Indeed, following [Kraus-Schroeter], we consider the more general left multiplication operator M_B defined by

$$M_B\rho = B\rho\,, \quad B \in \mathcal{B}(\mathrm{h})$$

which is bounded on $L_2(\mathrm{h})$, with norm $\|M_B\| = \|B\|$.

Regard the isomorphism $\mathcal{V}$ from $L_2(\mathrm{h})$ onto $\mathrm{h} \otimes \mathrm{h}$ defined by

$$\mathcal{V}\,|u\rangle\langle v| = u \otimes \theta v\,,$$

which is extended by linearity and continuity to the whole space h, where θ is any anti-unitary operator on h such that $\theta^2 = \mathbb{1}$. One directly computes that

$$\mathcal{V} M_B \mathcal{V}^{-1} u \otimes v = \mathcal{V}\,|Bu\rangle\langle \theta v| = (B \otimes \mathbb{1})\, u \otimes v\,.$$

Thereby, it follows by linearity and density that

$$\mathcal{V} M_B \mathcal{V}^{-1} = B \otimes \mathbb{1}\,.$$

Hence, we can identify $L_2(\mathrm{h})$ with $\mathrm{h} \otimes \mathrm{h}$ and consider the operator $B \mapsto B \otimes \mathbb{1}$ instead of M_B.

If $U(t)$ is a strongly continuous unitary group on h and A is the corresponding unbounded selfadjoint generator, with associated spectral measure $(E_\lambda)_{\lambda\in\mathbb{R}}$. Then

$$U(t) = \int e^{it\lambda} dE_\lambda \, .$$

The unitary group $U(t) \otimes \mathbb{1}$ and the spectral measure $(E_\lambda \otimes \mathbb{1})$ have the corresponding properties, in particular,

$$U(t) \otimes \mathbb{1} = \int e^{it\lambda} d(E_\lambda \otimes \mathbb{1}) \, .$$

Let $(\mathbb{U}(t))_{t\in\mathbb{R}}$ and $(\mathbb{E}_\lambda)_{\lambda\in\mathbb{R}}$ be the corresponding unitary group and spectral family on $L_2(\mathrm{h})$, such that

$$\mathcal{V}\mathbb{U}_t\mathcal{V}^{-1} = U(t)\otimes \mathbb{1}\,, \quad \forall\, t\in\mathbb{R}\,.$$

Consider $\mathbb{A}$ the corresponding selfadjoint generator and the representations

$$\mathbb{A} = \int \lambda d\mathbb{E}_\lambda\,, \quad \mathbb{U}_t = \int e^{it\lambda} d\mathbb{E}_\lambda\,,$$

whence if A is positive, then so is $\mathbb{A}$. From [Lemma 2, of Kraus-Schroeter], the explicit action of $\mathbb{A}$ is

$$\mathbb{A}\eta = A\eta, \qquad \text{for all } \eta\in\mathrm{dom}\,(\mathbb{A}) = \{\eta\in \mathrm{L}_2(\mathrm{h})\ :\ \mathrm{A}\eta\in\mathrm{L}_2(\mathrm{h})\}\,. \tag{2}$$

We will freely use spectral functional calculus for unbounded selfadjoint operators.

Lemma

A positive selfadjoint operator A is integrable with respect to a state ρ if and only if $A^{\frac{1}{2}}\rho^{\frac{1}{2}} \in L_2(\mathrm{h})$. In such a case, one has that:

$$\mathrm{tr}\,(\rho A) = \int \lambda d \left\langle \rho^{\frac{1}{2}}, \mathbb{E}\rho^{\frac{1}{2}} \right\rangle_2 = \left\langle A^{\frac{1}{2}}\rho^{\frac{1}{2}}, A^{\frac{1}{2}}\rho^{\frac{1}{2}} \right\rangle_2 , \tag{3}$$

where $\mathbb{E}$ is the spectral measure of $\mathbb{A}$. Besides, for the spectral decomposition $\rho = \sum_{k\in\mathbb{N}} \rho_k |u_k\rangle\langle u_k|$, equation (3) turns into

$$\mathrm{tr}\,(\rho A) = \sum_{k\in\mathbb{N}} \rho_k \left\| A^{\frac{1}{2}} u_k \right\|^2 .$$

Sketch of the proof:
If ρ is A-traceable then for a unit vector $h \in \mathrm{h}$,

$$\left\langle \rho^{\frac{1}{2}} h, A_\epsilon \rho^{\frac{1}{2}} h \right\rangle \leq \operatorname{tr}\left(\rho^{\frac{1}{2}} A_\epsilon \rho^{\frac{1}{2}} \right) = \operatorname{tr}\left(\rho A_\epsilon \right) < \infty$$

for all $\epsilon > 0$. Thereby, $\left\langle \rho^{\frac{1}{2}} h, A_\epsilon \rho^{\frac{1}{2}} h \right\rangle$ increases as $\epsilon \to 0$, to the finite value $\|A^{\frac{1}{2}} \rho^{\frac{1}{2}} h\|^2$ and one obtains that $\operatorname{ran} \rho^{\frac{1}{2}} \subset \operatorname{dom} \mathrm{A}^{\frac{1}{2}}$. Besides, if $\{u_k\}_{k\in\mathbb{N}}$ is an orthonormal basis, by the monotone convergence theorem,

$$\left\| A^{\frac{1}{2}} \rho^{\frac{1}{2}} \right\|_2^2 = \sum_{k\in\mathbb{N}} \left\| A^{\frac{1}{2}} \rho^{\frac{1}{2}} u_k \right\|^2 = \lim_{\epsilon\to 0} \sum_{k\in\mathbb{N}} \left\langle \rho^{\frac{1}{2}} u_k, A_\epsilon \rho^{\frac{1}{2}} u_k \right\rangle = \operatorname{tr}\left(\rho A \right) < \infty \,. \quad (4)$$

Hence, $A^{\frac{1}{2}} \rho^{\frac{1}{2}} \in L_2(\mathrm{h})$.

Conversely, if $A^{\frac{1}{2}}\rho^{\frac{1}{2}} \in L_2(\mathrm{h})$ then $\rho^{\frac{1}{2}} \in \operatorname{dom} \mathbb{A}^{\frac{1}{2}}$, i.e., $\int \lambda d \left\langle \rho^{\frac{1}{2}}, \mathbb{E}\rho^{\frac{1}{2}} \right\rangle_2 < \infty$. Besides,

$$\lambda(1+\epsilon\lambda)^{-1} \leq \lambda\,, \quad \epsilon, \lambda \geq 0\,.$$

Thus, by Lebesgue's Theorem on Dominated Convergence,

$$\begin{aligned} \operatorname{tr}(\rho A) &= \lim_{\epsilon\to 0} \left\langle \rho^{\frac{1}{2}}, A(I-\epsilon A)^{-1}\rho^{\frac{1}{2}} \right\rangle_2 \\ &= \lim_{\epsilon\to 0} \int \frac{\lambda}{1+\epsilon\lambda} d\left\langle \rho^{\frac{1}{2}}, \mathbb{E}\rho^{\frac{1}{2}} \right\rangle_2 = \int \lambda d \left\langle \rho^{\frac{1}{2}}, \mathbb{E}\rho^{\frac{1}{2}} \right\rangle_2 , \end{aligned} \tag{5}$$

as required. Equalities (3) and (**??**) are straightforward from (4) and (5).

Theorem

Let ρ be a state, A a selfadjoint operator and $n \geq 0$. If (A^n) is ρ-integrable then

$$\left\langle A^n \right\rangle_\rho = \mathrm{tr}\left(\rho A^n\right) = i^{-n} \frac{d^n}{dt^n} \mathrm{tr}\left(\rho e^{itA}\right) 1_{t=0} \ .$$

Remark: For $n \geq 0$ and a Gaussian state ρ, if $(p(z)^n)$ is ρ-integrable, then Theorem 3 implies that the n-th moment of the field operator (Weyl moment) in ρ satisfies

$$\left\langle p(z)^n \right\rangle_\rho = \mathrm{tr}\left(\rho p(z)^n\right) .$$

Indeed, we have the following

Corollary

The n-th moment of the field operator $p(z)$ in a Gaussian state $\rho = \rho(\omega, S)$ is given explicitly by

$$\left\langle p(z)^n \right\rangle_\rho = (-i)^n \frac{d^n}{dt^n} e^{it(\omega,z) - \frac{1}{2}t^2(z,Sz)} \rceil_{t=0} \ .$$

Therefore, denoting by $\left\langle \left(2p(z) - (w,z)\right)^n \right\rangle_\rho$ the n-th **centered** moment of $p(z)$ in ρ, the following recurrence relation holds:

$$\left\langle \left(2p(z) - (w,z)I\right)^n \right\rangle_\rho = \begin{cases} 0\,, & \text{for } n \text{ odd} \\ (z,Sz)^{\frac{n}{2}}(n-1)!!\,, & \text{for } n \text{ even} \end{cases}$$

This motivates the following combinatorial approach to Gaussianity.

[Luigi Accardi, Generalized gaussianity: states and semigroups, talk given at UAM-Iztapalapa, September 2022.]

Allows to include other classes of Gaussianity: Fermionic-Gaussianity, Free-Gaussianity, Monotone-Gaussianity,...

Gaussian maps on algebras

In this section, the term algebra means a complex associative algebra with identity.

Definition

A **pair partition** of the **ordered set** $\{1, \ldots, 2p\}$ ($p \in \mathbb{N}$) is a set of pairs $\{i_h, j_h\}_{h=1}^{p}$ such that

$$\begin{cases} \{i_k, j_k\}_{k=1}^{p} \text{ is a partition of } \{1, \ldots, 2p\} \\ i_k < j_k \ , \ \forall k = 1, \ldots, p \end{cases} \tag{6}$$

Let $\mathcal{B}$ be a topological algebra. A sub–set $B \subseteq \mathcal{B}$ is called a set of generators if the linear span of products of elements of B is dense in $\mathcal{B}$

Definition

Let $\mathcal{B}$ and $\mathcal{C}$ be algebras with $\mathcal{C}$ commutative and let $E : \mathcal{B} \longrightarrow \mathcal{C}$ be a linear map. Given a set of generators $B \subseteq \mathcal{B}$ such that for any $b_1, \ldots, b_n \in B$,

$$E(b_1 \cdot \cdots \cdot b_n) = 0 \text{ if } n \text{ is odd} \tag{7}$$

and, for each even $n = 2p \in \mathbb{N}$

$$E(b_1 \cdots b_{2p}) = \frac{1}{p!} \sum_{(i_1,j_1;\, i_2,j_2;\, \ldots\, ;i_p,j_p) \in PP(2p)} E(b_{i_1} b_{j_1}) \cdots E(b_{i_p} b_{j_p}) \tag{8}$$

then E is called a (mean zero) *boson-gaussian map* on $\mathcal{B}$.

Definition

A **combinatorial boson Gaussian state** is a pair $(\mathcal{B}, E)$, where $\mathcal{B}$ is an algebra with a notion of positivity (C^*-algebra) and E is a complex-valued (i.e., $C = \mathbb{C}$) *positive and identity preserving* Gaussian map on $\mathcal{B}$, i.e.

$$E(1_{\mathcal{B}}) = 1_{\mathbb{C}}$$

Example: The case when the set B consists of a single element: $B = \{b\}$

In this case, setting

$$b_1 = b_2 = \cdots = b_{2p} =: b \tag{9}$$

in the identity for the mixed moments

$$E(b_1 \cdots b_{2p}) = \sum_{(i_1,j_1;\, i_2,j_2;\, \dots;\, i_p,j_p) \in PP(2p)} E(b_{i_1} b_{j_1}) \cdots E(b_{i_p} b_{j_p}) \tag{10}$$

one finds

$$\begin{aligned} E(b^{2p}) &= \sum_{(i_1,j_1;\, i_2,j_2;\, \dots;\, i_p,j_p) \in PP(2p)} E(b^2)^p = |PP\,(2p)|\; E(b^2)^p \\ &= \frac{(2p)!}{2^p p!}\, E(b^2)^p = (2p-1)!!\, E(b^2)^p, \qquad \forall p \in \mathbb{N} \end{aligned} \tag{11}$$

which are the moments of a boson gaussian random variable with variance $E(b^2)$.

The set of all combinatorial Gaussian states is denoted by $\mathcal{E}_G$. The quantity

$$q(b_1, b_2) := E(b_1 \cdot b_2)$$

is called the E–covariance (or E–2-point function) of the family B. The expectation values $E(b_1 \cdots b_{2p})$ are called the *generalized mixed E–moments* of B (or *correlators*, or *correlation functions*).

This section aims at proving the equivalence of the combinatorial and the analytical approaches to gaussianity. To do so we shall introduce the notion of **combinatorial Gaussian state field** and prove that this notion is equivalent with the concept of analytic Gaussian state.

Combinatorial Gaussian state fields

Definition

A combinatorial Gaussian state field (combinatorial Gaussian field, for short) on a Hilbert space h is a function $z \mapsto (\mathcal{B}_z, E)$ from h into $\mathcal{E}_G$, with $\mathcal{B}_z$ being a sub-algebra of a fixed algebra $\mathcal{B}$ such that

(i) The E-mean function $w(z) := E\big(b(z)\big)$ is a bounded real linear functional on h.

(ii) The E-covariance function $\mathcal{S}(z,u) := E\big(b(z)b(u)\big)$ is a bounded positive real symmetric bilinear form on $\mathrm{h} \times \mathrm{h}$, i.e., it is real linear in both variables and

- *(i)* $\mathcal{S}(z,u) = \mathcal{S}(u,z), \ \forall\ z,u \in \mathrm{h}$ and
- *(ii)* $\mathcal{S}(z,z) \geq 0, \ \ \forall \ \ z \in \mathrm{h}$

Remark

(i) By Riesz's theorem there exits a mean vector in h also denoted by w such that

$$w(z) = (w, z)$$

(ii) Notice that the element $b(z) - E\big(b(z)\big)$ is another generator of the algebra generated by $b(z)$, hence, by replacing w with $\hat{w}(z) = E\big(b(z) - E(b(z))\big),\ \forall\ z \in \mathrm{h}$ if necessary, we can assume that $w(z) = 0,\ \forall\ z \in \mathrm{h}$, i.e., E is a zero-mean (or centered) field.

(iii) Clearly the bilinear form $E\big(b(z)b(u)\big)$ is symmetric if the generators $b(z)$ and $b(u)$ commute for all $z, u \in \mathrm{h}$.

(iv) Since the bilinear form $\mathcal{S}$ is bounded, there exists a bounded, real symmetric operator S, called the covariance operator of the combinatorial Gaussian field, such that

$$\mathcal{S}(z, u) = (z, Su),\ \forall\ z, u \in \mathrm{h}$$

.

Example: (The case of one single generator)
Assume that for each $z \in \mathbb{C}$, we have a combinatorial boson Gaussian state $(\mathcal{B}_z, E)$ with $\mathcal{B}_z$ generated by a single element $b(z)$, and we have for each $z \in \mathbb{C}$,

$$\begin{aligned} E(b^{2p}(z)) =& \sum_{(i_1,j_1;\, i_2,j_2;\, \dots;\, i_p,j_p) \in PP(2p)} E(b^2(z))^p = |PP\,(2p)|\; E(b^2(z))^p \\ =& \frac{(2p)!}{2^p p!}\, E(b^2(z))^p = (2p-1)!!\, E(b^2(z))^p, \qquad \forall p \in \mathbb{N} \end{aligned} \tag{12}$$

The algebra $\mathcal{B}_z$ generated by $p(z)$ coincides with the algebra generated by the shifted field operator $p(z) + cI, \;\; c \in \mathbb{C}$.

Lemma

Let $\rho = \rho(w, S)$ be an analytic Gaussian state and for each $z \in \mathbb{C}$, let $\mathcal{B}_z$ be the algebra generated by the field operator $p(z)$. Under our assumptions, the state ρ induces a combinatorial boson Gaussian field $(\mathcal{B}_z, E_\rho)$ with $E_\rho : \mathcal{B}_z \mapsto \mathbb{C}$ defined by means of

$$E_\rho(b(z)) := tr(\rho b(z)), \;\; b(z) \in \mathcal{B}_z \tag{13}$$

We already proved that if ρ is analytic Gaussian, then $(\mathcal{B}_z, E_\rho)$ is a combinatorial boson Gaussian field. Indeed,

$$E_\rho\Big(b(z)^n\Big) = \Big\langle\Big(2p(z) - (w,z)I\Big)^n\Big\rangle_\rho = \begin{cases} 0\,, & \text{for } n \text{ odd} \\ (z, Sz)^{\frac{n}{2}}(n-1)!!\,, & \text{for } n \text{ even} \end{cases}$$

and $(z, Sz)^{\frac{n}{2}} = E_\rho\Big(b(z)\Big)^{\frac{n}{2}}$.

Theorem

A state ρ is analytic Gaussian if and only if the corresponding pair $(\mathcal{B}_z, E_\rho)$ is a combinatorial boson Gaussian field.

Proof. It suffices to prove that any combinatorial boson Gaussian field $z \mapsto (\mathcal{B}_z, E)$ of the form (13) is induced by an analytic Gaussian state.
Let $\mathbb{E}$ be the spectral measure of $p(z)$ on $L_2(\mathrm{h})$. By the previous Theorem 3, Lebesgue's Theorem on Dominated Convergence and the combinatorial recurrence, one has that

$$\begin{aligned} e^{-(w,z)t}\sum_{n\geq 0}\frac{1}{n!}\mathrm{tr}\Big(\rho\big(p(z)\big)^n\Big)t^n &= \int e^{-t(w,z)+2t\lambda}d\left\langle \rho^{\frac{1}{2}},\mathbb{E}\rho^{\frac{1}{2}}\right\rangle_2 \\ &= \sum_{n\geq 0}\frac{1}{n!}\mathrm{tr}\Big(\rho\big(p(z)-(w,z)I\big)^n\Big)t^n \\ &= \sum_{n\geq 0}\frac{1}{(2n)!}(z,Sz)^n(2n-1)!!t^{2n} \\ &= \sum_{n\geq 0}\frac{1}{n!}\Big(\frac{(z,Sz)}{2}\Big)^n t^{2n} = e^{\frac{1}{2}(z,Sz)t^2}, \end{aligned} \tag{14}$$

since $(2n-1)!! = (2n)!(2^n n!)^{-1}$.

Then, the moment-generating function of the observable $p(z)$ in ρ satisfies

$$g_{\rho,p(z)}(t) = e^{(w,z)t+\frac{1}{2}(z,Sz)t^2}, \quad t \in \mathbb{R}.$$

Replacing t by $-i$, we get

$$\mathrm{tr}\,(\rho W_z) = \sum_{n\geq 0} \frac{1}{n!}\mathrm{tr}\Big(\rho\big(ip(z)\big)^n\Big) = e^{-i(w,z)-\frac{1}{2}(z,Sz)}$$

Consequently, ρ is an analytic Gaussian state.

Vacuum state

For the vacuum state $\Omega = |\mathbb{1}\rangle\langle\mathbb{1}|$, where $\mathbb{1}$ is the ground state that belongs to $\mathrm{dom}\, p(z)^n$ for all $n \geq 1$, we have

$$\left\langle (2p(z))^n \right\rangle_\Omega = 2^n \langle \mathbb{1}, p(z)^n \mathbb{1} \rangle = \begin{cases} 0\,, & \text{for odd } n \\ |z|^n\,(n-1)!!\,, & \text{for even } n \end{cases}$$

So that, Ω is Gaussian, $\Omega = \Omega(0, I)$.

Gibbs state at inverse temperature $\beta > 0$

The Gibbs state $\rho_\beta = (1 - e^{-\beta})e^{-\beta a^\dagger a}$ satisfies

$$\left\langle (2p(z))^n \right\rangle_{\rho_\beta} = \begin{cases} 0\,, & \text{for } n \text{ odd} \\ |z|^n \left(\coth\left(\beta/2\right)\right)^{\frac{n}{2}} (n-1)!!\,, & \text{for } n \text{ even} \end{cases}$$

So that, ρ_β is Gaussian, indeed $\rho_\beta = \rho_\beta(0, \coth\left(\beta/2\right) I)$.

Quantum channels

A completely positive trace preserving map T acting on $\mathcal{B}(\mathrm{h})$ is a quantum *Gaussian channel* if for any initial Gaussian state ρ the output state $T(\rho)$ is also Gaussian.

One can produce more examples of Gaussian states using quantum *Gaussian channels*.

Quantum channels

For L unitary, A selfadjoint and any A-traceable state ρ, it follows that

$$\mathrm{tr}\left(L\rho L^{*}A\right) = \mathrm{tr}\left(\rho L^{*}AL\right) . \tag{15}$$

Coherent channel

For $u \in \mathbb{C}$, the coherent channel $T_u(\rho) = W_u \rho W_u^*$ is Gaussian. We first note that $p(z) = \frac{i}{2}\frac{d}{dt} W_{tz}\, 1_{t=0}$ and

$$\begin{aligned} p(z)W_u &= \frac{i}{2}\frac{d}{dt} W_{tz} W_u\, 1_{t=0} = \frac{i}{2}\frac{d}{dt} e^{-2itp(z)} W_u W_{tz}\, 1_{t=0} \\ &= W_u\,(p(z) + \sigma(z,u)) = W_u\,(p(z) - (iu,z))\ , \end{aligned} \tag{16}$$

where we used that $\sigma(z,u) = -(iu,z)$.

Thereby, it is easy to compute by the binomial formula that

$$(2p(z) - (w - 2iu, z)I)^n\, W_u = W_u\,(2p(z) - (w,z)I)^n\ , \quad n \in \mathbb{N}\,. \tag{17}$$

Thus, for a Gaussian state ρ that is traceable for $(p(z)^n)_\pm$, one has by virtue of (15) and (17) that

$$\mathrm{tr}\Big(T_u(\rho)\,(2p(z) - (w - 2iu, z)I)^n\Big) = \Big\langle(2p(z) - (w, z)I)^n\Big\rangle_\rho ,$$

i.e., then $(p(z)^n)_\pm$ is $T_u(\rho)$-integrable and satisfies conditions of our Theorem. Hence, it is Gaussian with mean value $w - 2iu$ and covariance S.

Coherent states

In particular, if ρ is the vacuum state, then the coherent state $|\psi_u\rangle\langle\psi_u| = T_u(|\mathbb{1}\rangle\langle\mathbb{1}|)$ is Gaussian with mean value vector $-2iu$ and covariance matrix I.

Squeezing channel

The squeezing channel is given by $\mathcal{S}_\zeta(\rho) = S_\zeta \rho S_\zeta^*$, with $\zeta \in \mathbb{C}$. The *squeezing* operator given by

$$S_\zeta := e^{\frac{1}{2}(\zeta a^{\dagger 2} - \bar{\zeta} a^2)}, \qquad \zeta \in \mathbb{C}$$

which is unitary and satisfies $S_\zeta^* = S_{-\zeta}$.

For any Gaussian state ρ which is traceable for $(p(z)^n)_\pm$, $n \geq 0$, it follows by (15), (**??**) and the binomial formula that

$$\begin{aligned}\operatorname{tr}\left(\mathcal{S}_\zeta(\rho)\left(2p(z) - (U_\zeta w, z)I\right)^n\right) &= \operatorname{tr}\left(\rho S_\zeta^*\left(2p(z) - (w, U_\zeta z)I\right)^n S_\zeta\right) \\ &= \operatorname{tr}\left(\rho\left(2p(U_\zeta z) - (w, U_\zeta z)I\right)^n\right) \\ &= \begin{cases} 0\,, & \text{for } n \text{ odd} \\ (z, U_\zeta S U_\zeta z)^{\frac{n}{2}}(n-1)!!\,, & \text{for } n \text{ even}\end{cases}\end{aligned}$$

where U_ζ is the invertible, symmetric and real matrix

$$U_\zeta := \begin{pmatrix}\cosh r - \sinh r\cos\theta & -\sinh r\sin\theta \\ -\sinh r\sin\theta & \cosh r + \sinh r\cos\theta\end{pmatrix} \in \mathcal{B}_{\mathbb{R}}(\mathbb{C})$$

and $U_\zeta z = z\cosh r - \bar{z}e^{i\theta}\sinh r$.

Therefore, $\mathcal{S}_\zeta(\rho)$ is $(p(z)^n)_\pm$- traceable and one infers by virtue of our Theorem that it is Gaussian with mean value vector $U_\zeta w$ and covariance matrix $U_\zeta S U_\zeta$, where S is the covariance of the initial state.

Squeezed states

The squeezed coherent state is given by

$$\rho_s = S_\zeta W_u \, |\mathbb{1}\rangle\langle\mathbb{1}| \, W_u^* S_\zeta^* = \mathcal{S}_\zeta \left(T_u(|\mathbb{1}\rangle\langle\mathbb{1}|) \right) , \quad (\zeta, u \in \mathbb{C})$$

which is Gaussian with mean value vector $-2U_\zeta iu$ and covariance matrix U_ζ^2.

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Thank you for your attention!