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On "Problems on von Neumann Algebras by R. Kadison, 1967"

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Abstract A brief summary of the development on Kadison's famous problems (1967) is given. A new set of problems in von Neumann algebras is listed.

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1 Introduction

Thirty-six years after the Baton Rouge conference on operator algebras, the subject has undergone vast development. Two manuscripts by R. Powers and M. Tomita, respectively, distributed at the meeting changed the face of the subject. There is no doubt that Tomita–Takesaki's modular theory for von Neumann algebras is one of the greatest achievements in mathematics in the last century. Another manuscript consisting of twenty problems on von Neumann algebras by R. Kadison was also available at the meeting. The manuscript has never been published but is well known in the operator algebra community. The studies on those problems have influenced the subject tremendously and the remaining unsolved problems are still the focus of research nowadays.

In this article, we shall give an overview on the current status of R. Kadison's famous list of "Problems on Von Neumann Algebras" (1967). Many of the problems have a long history and interesting stories are involved. Even some of the solutions have twists and untold stories. To avoid any non-accuracy, we shall not include any comments and discussions to the solutions. The second part of the paper consists of a list of open problems. Some background of the new set of problems was explained at talks by the author at GPOTS (2002), a summer school at Fudan University in Shanghai (2002) and on some other occasions. Many of those problems are well-known open questions in this subject. We have not included any of the origins of the problems in this note. We hope to write a more extensive article on the same topic soon.

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2 Kadison's Problems

Here we quote the statements Kadison wrote in 1967 (before stating his problems):

The following set of problems is presented without an attempt to establish sources. Many of them stem from the Murray–von Neumann papers; and many are too fundamental to be really attributable to one source. Some of them have appeared (and reappeared) in my own work. I am tempted to discuss their origins, consequences and interrelations but refrain because this would be tantamount (if done thoroughly) to publishing incomplete papers. For better or worse, this is against accepted mathematical research practice. Perhaps I have moved too far to the other side—avoiding all discussion.

In re-examining the list, I note that they are all "yes" or "no" questions (apparently an attempt to keep them "crisp"). Of course, such problems as "Find isomorphism invariants for factors of type II_1*" are implicit in many of the questions. I would hope that the listed questions serve to stimulate attempts at theories answering the more general problems. Answers to several of the questions would yield technical lemmas of great use. Each of the questions seems to me to require some technique not presently available and very likely valuable.*

1 *Are there more than three isomorphism classes of factors of type* II_1*? Are there a countable number of such classes? Are there an uncountable number?*

The answers to all three questions are "yes!" The following people have contributed to the solutions: (the first two are due to Murray and von Neumann, 1936, Rings I; J. T. Schwartz, 1963, has the third example;) W. M. Ching, 1969 (4th); S. Sakai, 1968/69, J. Dixmier and E. C. Lance, 1969, G. Zeller-Meier, 1969 (5th–10th); D. McDuff, 1969 (Ching and Sakai, 1969–1970) (uncountably infinitely many).

2 *The weak-operator closure of the left regular representation of the free (non-abelian) group on two or more generators is a factor of type* II_1*. Are these factors isomorphic for different numbers of generators?*

This question is still open! In recent years, D. Voiculescu and many others have made progresses toward its solution. Now we know that they are either all isomorphic to one another or all distinct.

3 *The algebra of* $n \times n$ *matrices with entries in a given factor of type* II_1 *is, again, a factor of type* II_1*. Is it isomorphic to the original factor of type* II_1*? Is this the case for* $n \times n$ *matrices over the* II_1 *factor arising from the free group on* m *generators?*

The answer to the first question is "no" by Popa, 2001. The second question is equivalent to Problem 2 by D. Voiculescu, 1991.

4 *If* $\mathcal{M}$ *is a factor of type* II_1 *with commutant* $\mathcal{M}'$ *and* x_0 *is a (joint) trace vector for both* $\mathcal{M}$ *and* $\mathcal{M}'$*, for each* A *in* $\mathcal{M}$*, there is a unique* A' *in* $\mathcal{M}'$ *such that* $Ax_0 = A'x_0$*. The mapping* $A \to A'$ *is a* $*$ *anti-isomorphism* $((AB)' = B'A')$ *of* $\mathcal{M}$ *onto* $\mathcal{M}'$*. Are* $\mathcal{M}$ *and* $\mathcal{M}'$ $*$ *isomorphic?*

The answer is "no" by A. Connes, 1975.

5 *With $\mathcal{M}$ and $\mathcal{N}$ von Neumann algebras, is* $(\mathcal{M}\,\bar{\otimes}\,\mathcal{N})' = \mathcal{M}'\,\bar{\otimes}\,\mathcal{N}'$?

The answer is "yes!" It is one of the consequences of the Tomita-Takesaki theory, 1970.

6 *Does the hyperfinite factor of type* II_1 *contain a non-hyperfinite factor of type* II_1 ? *Is it the tensor product of two factors at least one of which is not hyperfinite?*

The answers to both questions are "no" by A. Connes, 1976.

7 *Does each self-adjoint operator in a* II_1 *factor lie in some hyperfinite subfactor?*

The answer is "no!"; S. Popa (1983) showed that the self-adjoint abelian subalgebra of a free group factor generated by any of the unitary operators corresponding to any of the free group generators is not contained in a hyperfinite subfactor.

8 *Can a non-scalar operator in a* II_1 *factor commute with some maximal hyperfinite subfactor?*

Yes!—Popa, 1983.

9 *Is a factor $\mathcal{M}$ hyperfinite* (*the weak-operator closure of an ascending union of self-adjoint subalgebras each containing the identity operator and each isomorphic to some full finite-dimensional matrix algebra*) *if* $\{a_1U_1TU_1^* + \cdots + a_nU_nTU_n^* : \sum_j a_j = 1, a_j \geq 0\}$ *has a weak-operator closure $\mathcal{S}$ with $\mathcal{S} \cap \mathcal{M}'$ non-null for every bounded operator T? Is it hyperfinite if it is the weak-operator closure of the left regular representation of an amenable group? of a* (*discrete*) *solvable group?*

Yes!—to all. This is mostly due to A. Connes, 1976.

10 *Does every factor $\mathcal{M}$ of type* III *contain a von Neumann subalgebra $\mathcal{R}$ for which* $\mathcal{R} \neq (\mathcal{R}' \cap \mathcal{M})' \cap \mathcal{M}$? *Is there such an $\mathcal{R}$ contained properly in $\mathcal{M}$ for which $\mathcal{R}' \cap \mathcal{M}$ consists of scalars?*

Yes!—to both questions by M. Takesaki (and A. Connes), 1973. A stronger result was proved by Longo and Popa (1984): if $\mathcal{M}$ is a type III factor with a separable predual, then $\mathcal{M}$ contains a *hyperfinite* (proper) subfactor $\mathcal{R}$ with $\mathcal{R}' \cap \mathcal{M} = \mathbf{C}I$.

11 *If $\mathcal{N}$ is a subfactor of the factor $\mathcal{M}$ for which $\mathcal{N}' \cap \mathcal{M}$ consists of scalars, will some maximal abelian subalgebra of $\mathcal{N}$ be a maximal abelian subalgebra of $\mathcal{M}$?*

Yes—if there is a normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ by Popa, 1983. In general, the answer is "no" by Popa and Ge, 1998.

12 *If $\mathcal{N}$ is a subfactor of the factor $\mathcal{M}$ and each maximal abelian subalgebra of $\mathcal{N}$ is maximal abelian in $\mathcal{M}$, is $\mathcal{N} = \mathcal{M}$?*

The answer is "no!" An example of a type III factor was constructed by Takesaki, 1973; the type II_1 case was due to Popa, 1983.

13 *Is each factor of type* II_1 *isomorphic to the weak-operator closure of the left regular representation of some discrete group?—to the factors arising from a group acting on a measure space? For such factors, there is a complete orthonormal basis for the Hilbert space consisting of trace vectors—is there such a basis for all* II_1 *factors?*

The answer to the first is "no" (see Problem 4); and "no" to the second by D. Voiculescu, 1994. The last is still open.

14 *Is each factor generated by two self-adjoint operators?—each von Neumann algebra?—the factor arising from the free group on three generators?—is each von Neumann algebra finitely generated?*

All are open. Partial answers have been obtained by many.

15 *If a self-adjoint operator algebra is finitely generated* (*algebraically*) *and each self-adjoint operator in it has finite spectrum, is it finite-dimensional?*

It is still open (the assumption is that the algebra may not be norm closed).

16 *If* φ *is the* * *anti-isomorphism of the factor* $\mathcal{M}$ *of type* II_1 *onto its commutant* $\mathcal{M}'$ *arising from a joint trace vector* x_0 (*see Problem* 4) *and* $\mathcal{A}$ *is a maximal abelian subalgebra of* $\mathcal{M}$ *such that* $\mathcal{A}$ *and* $\varphi(\mathcal{A})$ *generate a maximal abelian subalgebra of the algebra* $\mathcal{B}(\mathcal{H})$ *of all bounded operators on* $\mathcal{H}$*, we say that* $\mathcal{A}$ *is a* simple maximal abelian subalgebra *of* $\mathcal{M}$*. Does each factor of type* II_1 *possess a simple maximal abelian subalgebra?—Does the factor arising from the free group on two generators?*

No—to both by Ge, 1997.

17 *If a C*-algebra acts on a Hilbert space so that each of its maximal abelian subalgebras is weak-operator closed, is the C*-algebra weak-operator closed?*

The answer is "yes" by G. Pedersen, 1972.

18 *If* $\mathcal{F}$ *is a family of bounded self-adjoint operators, let* $\mathcal{F}^{\mathrm{m}}$ *and* $\mathcal{F}_{\mathrm{m}}$ *be the sets of weak-operator limits of bounded monotone increasing and decreasing nets in* $\mathcal{F}$*, respectively. With* $\mathcal{A}_{\mathrm{sa}}$ *the self-adjoint operators in a C*-algebra* $\mathcal{A}$*, is* $(\mathcal{A}_{\mathrm{sa}}^{\mathrm{m}})_{\mathrm{m}} = \mathcal{A}_{\mathrm{sa}}''$*?—Is* $(\cdots((\mathcal{A}_{\mathrm{sa}}^{\mathrm{m}})_{\mathrm{m}})^{\mathrm{m}}\cdots)_{\mathrm{m}} = \mathcal{A}_{\mathrm{sa}}''$ (*finite application of the process of taking limits of monotone nets*)*?*

Yes—due to G. Pedersen, 1971. (No—to the first question with a non-separable assumption on $\mathcal{A}$ by C. Akemann, 1969.)

19 *Is each countably additive, non-negative real-valued function defined on the projections in a von Neumann algebra* $\mathcal{R}$ *and* 1 *at the identity operator the restriction of a state on* $\mathcal{R}$ (*separable Hilbert space*)*?—when* $\mathcal{R}$ *is a factor of type* II_1*?*

The answers are "yes!". (The type I case is due to A. Gleason, 1957.) The cases of type II_∞ and type III are due to E. Christensen, 1982. The following people made contributions to the type II_1 case (1983–1985): F. Yeadon, M. Matveichuk, A. Paszkiewicz and U. Haagerup.

20 *Is each pure state of $\mathcal{B}(\mathcal{H})$ multiplicative on some maximal abelian subalgebra?*

This is still open!

3 Some Further Questions

We are not going to repeat the questions that remain open on the above list. But some of our problems may be related to the above solved or unsolved problems. Some of our problems have been around for sometime and others are relatively new. But it is very difficult to judge which ones are important in the course of the development of the subject. It has been clear that, after the intensive studies of injective factors and subfactors (or subalgebras) of the hyperfinite factor of type II_1, the next candidates in line are the free group factors. The isomorphism question and the generator problem for these factors are still central in the subject. The more detailed structural analysis of these factors will not only shed light onto the solution to these questions, but also bring more tools to the study of other factors. After careful examination of the above problems, one sees that many of the questions are concerned with how a known subalgebra, such as an abelian or injective algebra, is embedded into a factor. Thus the embedding of abelian algebras, injective algebras and other known classes of algebras into a factor as well as extensions by those known algebras will be important in the study of factors. Another important question is the search for a global description or a universal construction of all factors of type II_1. The last, probably most importantly, is the connection of the theory of von Neumann algebras (and operator algebras, in general) with other areas of mathematics and sciences.

Again, we keep all questions "crisp"—they are all "yes" or "no" questions.

1 Is every subfactor of a free group factor either an interpolated free group factor or an injective factor?

2 A subfactor (or subalgebra) is called *maximal* if it is not contained in any proper subalgebra other than itself. Can a non-hyperfinite factor of type II_1 have a hyperfinite subfactor as its maximal subfactor? Can a maximal subfactor of the hyperfinite factor of type II_1 have an infinite Jones index? Can $\mathcal{L}_{F_\infty}$ be embedded in $\mathcal{L}_{F_2}$ as a maximal subfactor?

3 Is there a factor with hyperfinite length n, $n \geq 3$?

4 Does every non-hyperfinite II_1 factor contain $\mathcal{L}_{F_2}$? Let $B(n,m)$, $n \geq 2, m \geq 665$ odd, be the free Burnside group with n generators and of order m. Does $\mathcal{L}_{B(n,m)}$ contain a free group subfactor?

5 Is $\mathcal{L}_{B(2,p)}$ isomorphic to $\mathcal{L}_{B(3,p)}$ for a large prime number p? Is $\mathcal{L}_{B(2,p)}$ prime? What is the fundamental group of $\mathcal{L}_{B(2,p)}$?

6 What is the free entropy dimension of $\mathcal{L}_{B(n,m)}$ for the standard unitary generators?

7 Are there infinitely many conjugacy classes of outer automorphisms on $\mathcal{L}_{F_2}$ of order n, for each $n \geq 2$?

8 Let F be the Thompson group (the group of all piecewise linear homeomorphisms of $[0,1]$ which, except at finitely many dyadic rational numbers, are differentiable with derivatives equal to powers of 2). It has a presentation

$$F = \langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leq i < n \rangle.$$

Let α be the (order two) automorphism of F defined by $\alpha(x_0) = x_0^{-1}, \alpha(x_1) = x_0 x_1 x_0^{-2}$. Then α extends to an automorphism of the group factor $\mathcal{L}_F$. Is α approximately inner? Is $\mathcal{L}_F$ hyperfinite?

9 Does every factor of type II_1 have an outer automorphism?

10 Is the free entropy dimension an invariant of the free group factor $\mathcal{L}_{F_2}$?

11 Can $\mathcal{L}_{B(n,m)}$ be embedded into the ultrapower of the hyperfinite II_1 factor? Is every factor of type II_1 embeddable into the ultrapower of the hyperfinite II_1 factor?

12 Let $\Lambda(\mathcal{M})$ be the Haagerup invariant of a factor of type II_1. Is $\Lambda(\mathcal{M} * \mathcal{N})$ equal to $\max\{\Lambda(\mathcal{M}), \Lambda(\mathcal{N})\}$, where $\mathcal{M} * \mathcal{N}$ is the reduced free product of $\mathcal{M}$ and $\mathcal{N}$?

13 Is $H^2(\mathcal{L}_{F_2}, \mathcal{L}_{F_2}) = 0$? Is $H^n(\mathcal{M}, \mathcal{M}) = 0$, for every von Neumann algebra $\mathcal{M}$ and each $n \geq 2$?

14 What are all the index values of subfactors $\mathcal{N}$ of the hyperfinite II_1 factor $\mathcal{R}$ with $\mathcal{N}' \cap \mathcal{R} = \mathbf{C}I$?

15 Is every bounded (non-self-adjoint) representation of $\mathcal{L}_{F_2}$ similar to a bounded representation? Is every bounded representation of any factor of type II_1 similar to a bounded representation?

16 Does every operator in a factor of type II_1 have a non-trivial invariant subspace so that the projection onto the subspace lies in the factor?

17 Is there a (non-self-adjoint) transitive subalgebra of $\mathcal{B}(\mathcal{H})$ so that it is not weak-operator dense in $\mathcal{B}(\mathcal{H})$?

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